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Field theory in 4D $\mathcal{N}=2$ conformally flat superspace

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Abstract: Building on the superspace formulation for four-dimensional $\mathcal{N}=2$ matter-coupled supergravity developed in [1], we elaborate upon a general setting for field theory in $\mathcal{N}=2$ conformally flat superspaces, and concentrate specifically on the case of anti-de Sitter (AdS) superspace. We demonstrate, in particular, that associated with the $\mathcal{N}=2$ AdS supergeometry is a unique vector multiplet such that the corresponding covariantly chiral field strength $\mathcal{W}_0$ is constant, $\mathcal{W}_0=1$. This multiplet proves to be intrinsic in the sense that it encodes all the information about the $\mathcal{N}=2$ AdS supergeometry in a conformally flat frame. Moreover, it emerges as a building block in the construction of various supersymmetric actions. Such a vector multiplet, which can be identified with one of the two compensators of $\mathcal{N}=2$ supergravity, also naturally occurs for arbitrary conformally flat superspaces. An explicit superspace reduction $\mathcal{N}=2\to\mathcal{N}=1$ is performed for the action principle in general conformally flat $\mathcal{N}=2$ backgrounds, and examples of such reduction are given.

Keywords: Extended Supersymmetry, Superspaces, Supergravity Models.

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Contents

1. Introduction

Recently, we have developed the superspace formulation for four-dimensional $\mathcal{N} = 2$ matter-coupled supergravity [1], extending the earlier construction for 5D $\mathcal{N} = 1$ supergravity [2, 3]. The locally supersymmetric action proposed in [1] has a striking similarity with the chiral action in 4D $\mathcal{N} = 1$ supergravity [4, 5] (see also [6, 7] for reviews). The action functional proposed in [1] can be written in the form:

$$S = \frac{1}{2\pi} \oint (u^+ \mathrm{d}u^+) \int \mathrm{d}^4x\, \mathrm{d}^4\theta \mathrm{d}^4\bar{\theta}\, \mathcal{E}\, \frac{\mathcal{L}^{++}}{\mathcal{S}^{++}\widetilde{\mathcal{S}}^{++}}\,, \qquad u_i^+ \mathcal{D}^i_\alpha \mathcal{L}^{++} = u_i^+ \bar{\mathcal{D}}^i_{\dot\alpha} \mathcal{L}^{++} = 0\,, \quad (1.1)$$

with $\mathcal{S}^{++}(u^+) := \mathcal{S}^{ij} u_i^+ u_j^+$ and $\widetilde{\mathcal{S}}^{++}(u^+) := \bar{\mathcal{S}}^{ij} u_i^+ u_j^+$. Here $\mathcal{E}^{-1} = \mathrm{Ber}(\mathcal{E}_{\underline{A}}{}^{\underline{M}})$, where $\mathcal{E}_{\underline{A}}{}^{\underline{M}}$ is the (inverse) vielbein appearing in the superspace covariant derivatives, $\mathcal{D}_{\underline{A}} = (\mathcal{D}_a, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\alpha}_i)$, and $\mathcal{S}^{ij}$ and $\bar{\mathcal{S}}^{ij}$ are special irreducible components of the torsion (see appendix A for more detail). The Lagrangian $\mathcal{L}^{++}(u^+)$ is a holomorphic homogeneous function of second degree with respect to auxiliary isotwistor variables $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$, which are introduced in addition to the superspace coordinates. The total measure in (1.1) includes a contour integral in the auxiliary isotwistor space.

Let us now recall the well-known chiral action [4, 5] in 4D $\mathcal{N} = 1$ old minimal ($n = -1/3$) supergravity [8, 9]:

$$S_{\text{chiral}} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \frac{L_{\rm c}}{R}\,, \qquad \bar{\nabla}_{\dot\alpha} L_{\rm c} = 0\,. \quad (1.2)$$

Here E^{-1} is the superdeterminant of the (inverse) vielbein $E_A{}^M$ that enters the corresponding superspace covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^{\dot\alpha})$, and R is the chiral scalar component of the torsion (following the notation of [7]). The action is generated by a covariantly chiral scalar Lagrangian $L_{\rm c}$.

The similarity between (1.1) and (1.2) is at least twofold. First of all, each action involves integration over the corresponding *full* superspace. Secondly, the Lagrangians in (1.1) and (1.2) obey covariant constraints which enforce $\mathcal{L}^{++}$ and $L_{\rm c}$ to depend on *half* of the corresponding superspace Grassmann variables. The latter property is of crucial importance. It indicates that there should exist a covariant way to rewrite each action as an integral over a submanifold of the full superspace such that the number of its fermionic directions is half of the number of such variables in the full superspace (i.e. two in the $\mathcal{N} = 1$ case and four if $\mathcal{N} = 2$). In the $\mathcal{N} = 1$ case, such a reformulation is well-known. Using the chiral supergravity prepotential [5], the action (1.2) can be rewritten as an integral over the chiral subspace of the curved superspace, see also [6, 7] for reviews (a somewhat more exotic scheme is presented in [10]). What about the $\mathcal{N} = 2$ case? There are numerous reasons to expect that the action (1.1) can be reformulated as an integral over an $\mathcal{N} = 1$ subspace of the curved $\mathcal{N} = 2$ superspace. In particular, this idea is natural from the point of view of the projective superspace approach [11, 12] to rigid $\mathcal{N} = 2$ superymmetric theories (the supergravity formulation given in [1] can be viewed to be a curved projective superspace). We hope to give a detailed elaboration of this proposal elsewhere.[1] Here we only provide partial

[1] In the 1980s, there appeared a series of papers [13, 14] devoted to projecting special off-shell $\mathcal{N} = 2$ supergravity theories into $\mathcal{N} = 1$ superspace. Specifically:

(i) refs. [13] dealt with the standard 40 + 40 formulation [15] for $\mathcal{N} = 2$ Poincaré supergravity realized in $\mathcal{N} = 2$ superspace [16, 17]; and

(ii) ref. [14] was concerned with $\mathcal{N} = 2$ conformal supergravity realized in $\mathcal{N} = 2$ superspace in [17].

supportive evidence by considering arbitrary conformally flat $\mathcal{N}=2$ superspaces, including a maximally symmetric supergravity background — 4D $\mathcal{N}=2$ anti-de Sitter superspace.

Unlike the case of simple anti-de Sitter supersymmetry (AdS) in four dimensions,[2] field theory in the $\mathcal{N}=2$ AdS superspace is practically terra incognita.[3] In the case of the $\mathcal{N}=2$ Poincaré supersymmetry in four dimensions, there exist two universal schemes to formulate general off-shell supersymmetric theories: the harmonic superspace [24, 25] and the projective superspace [11, 12]. To the best of our knowledge, no thorough analysis has been given in the literature regarding an extension of these approaches to the anti-de Sitter supersymmetry. One of the goals of the present paper is to fill this gap.

Before turning to the technical part of this paper, a comment is in order. The action (1.1) is equivalent to that originally given in [1]. The latter looks like

$$S=\frac{1}{2\pi}\oint(u^+\mathrm{d}u^+)\int\mathrm{d}^4x\,\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\mathcal{E}\,\frac{\mathcal{W}\bar{\mathcal{W}}\,\mathcal{L}^{++}}{(\Sigma^{++})^2}\,, \tag{1.3}$$

where $\mathcal{W}$ is the covariantly chiral field strength, $\bar{\mathcal{D}}^{\dot\alpha}_i\mathcal{W}=0$, of an Abelian vector multiplet such that $\mathcal{W}$ is everywhere non-vanishing, and

$$\Sigma^{++}(u^+):=\Sigma^{ij}u^+_iu^+_j\,,\quad \Sigma^{ij}=\frac{1}{4}\big(\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_\gamma+4\mathcal{S}^{ij}\big)\mathcal{W}=\frac{1}{4}\big(\bar{\mathcal{D}}^{(i}_{\dot\gamma}\bar{\mathcal{D}}^{j)\dot\gamma}+4\bar{\mathcal{S}}^{ij}\big)\bar{\mathcal{W}}\,. \tag{1.4}$$

Unlike (1.1), a notable feature of (1.3) is that it is manifestly super-Weyl invariant [1]. The $\mathcal{N}=1$ action (1.2) can also be rewritten in a manifestly super-Weyl invariant form:

$$S=\int\mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\frac{\bar\Psi\,\mathcal{L}_{\rm c}}{\Sigma}\,,\qquad \Sigma=-\frac{1}{4}\big(\bar\nabla^2-4R\big)\bar\Psi\,,\qquad \bar\nabla_{\dot\alpha}\Psi=0\,. \tag{1.5}$$

Here Ψ is a covariantly chiral scalar superfield required to be everywhere non-vanishing but otherwise arbitrary.

This paper is organized as follows. In section 2, after a brief review of the differential geometry of the 4D $\mathcal{N}=2$ AdS superspace, $\mathrm{AdS}^{4|8}$, we elucidate the structure of $\mathcal{N}=2$ AdS Killing supervectors, and then introduce projective supermultiplets living in $\mathrm{AdS}^{4|8}$. In section 3, the manifestly supersymmetric action in $\mathrm{AdS}^{4|8}$ is reduced to $\mathcal{N}=1$ superspace, and then several models for hypermultiplets, tensor and vector multiplets are considered. Section 4 begins with a general discussion of $\mathcal{N}=2$ conformally flat superspaces. We then

Since off-shell formulations for general matter couplings in $\mathcal{N}=2$ supergravity were not available at that time, applications of [13, 14] were rather limited. We hope that the progress achieved in [1] should revitalize the approaches pursued in [13, 14].

[2] The structural aspects of 4D $\mathcal{N}=1$ AdS superspace and corresponding field representations were thoroughly studied in [18] (see also [19, 20] for earlier work).

[3] The necessity of having an adequate superspace setting for $\mathcal{N}=2$ AdS supersymmetry became apparent in [21] where *off-shell* higher spin supermultiplets with $\mathcal{N}=2$ AdS supersymmetry were constructed. These $\mathcal{N}=2$ supermultiplets were realized in [21] as field theories in the $\mathcal{N}=1$ AdS superspace, by making use of the dually equivalent formulations for $\mathcal{N}=1$ supersymmetric higher spin theories previously developed in [22]. However, their off-shell $\mathcal{N}=2$ structure clearly hinted at the existence of a manifestly supersymmetric formulation in the $\mathcal{N}=2$ AdS superspace. Some progress toward constructing such a formulation has been made in [23].

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realize the $\mathcal{N}=2$ AdS superspace as locally conformal flat, work out the tropical prepotential for the intrinsic vector multiplet, and explicitly compute the $\mathcal{N}=2$ AdS Killing supervectors. In section 5, the action (1.1) in an arbitrary conformally flat $\mathcal{N}=2$ superspace is reduced to $\mathcal{N}=1$ superspace. As applications of this reduction, we consider several models for massive hypermultiplets in $\mathrm{AdS}^{4|8}$ and vector multiplets in the conformally flat superspace. Final comments and conclusions are given in section 6. The paper also contains four technical appendices. Appendix A is devoted to a short review of the superspace geometry of $\mathcal{N}=2$ conformal supergravity following [1]. In appendix B, we elaborate upon the projective-superspace description of Abelian vector multiplets in conformal supergravity (along with some properties previously presented in [1], new results are included in this appendix). Appendix C is devoted to a mini-review of the geometry of $\mathcal{N}=1$ AdS superspace and the corresponding Killing supervectors, following [7]. Finally, appendix D presents a summary of the stereographic projection for d-dimensional AdS spaces.

2. $\mathcal{N}=2$ anti-de Sitter supergeometry

The superspace geometry, which is quite compact to use and, at the same time, perfectly suitable to describe 4D $\mathcal{N}=2$ conformal supergravity and covariant projective matter supermultiplets, was presented in [1] (see appendix A for a concise review); its connection to Howe's formulation for conformal supergravity [17] is discussed in [1]. In such a setting, the 4D $\mathcal{N}=2$ AdS superspace

$$\mathrm{AdS}^{4|8}=\frac{\mathrm{OSp}(2|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(2)}$$

corresponds to a geometry with covariantly constant torsion:[4]

$$\mathcal{W}_{\alpha\beta}=\mathcal{Y}_{\alpha\beta}=0\,,\qquad \mathcal{G}_{\alpha\dot\beta}=0\,,\qquad \mathcal{D}^i_\alpha\mathcal{S}^{kl}=\bar{\mathcal{D}}^i_{\dot\alpha}\mathcal{S}^{kl}=0\,. \tag{2.1}$$

The integrability condition for these constraints is $[\mathcal{S},\mathcal{S}^\dagger]=0$, with $\mathcal{S}=(\mathcal{S}^i{}_j)$, and hence

$$\mathcal{S}^{ij}=q\,\boldsymbol{S}^{ij}\,,\qquad \overline{\boldsymbol{S}^{ij}}=\boldsymbol{S}_{ij}\,,\qquad |q|=1\,, \tag{2.2}$$

where q is a constant parameter. By applying a rigid $\mathrm{U}(1)$ phase transformation to the covariant derivatives, $\mathcal{D}^i_\alpha\to q^{-1/2}\mathcal{D}^i_\alpha$, one can set $q=1$. This choice will be assumed in what follows.

The covariant derivatives of the 4D $\mathcal{N}=2$ AdS superspace form the following algebra:

$$\{\mathcal{D}^i_\alpha,\mathcal{D}^j_\beta\}=4\boldsymbol{S}^{ij}M_{\alpha\beta}+2\varepsilon_{\alpha\beta}\varepsilon^{ij}\boldsymbol{S}^{kl}J_{kl}\,,\qquad \{\mathcal{D}^i_\alpha,\bar{\mathcal{D}}^{\dot\beta}_j\}=-2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c\,, \tag{2.3a}$$

$$[\mathcal{D}_a,\mathcal{D}^j_\beta]=\frac{\mathrm{i}}{2}(\sigma_a)_{\beta\dot\gamma}\boldsymbol{S}^{jk}\bar{\mathcal{D}}^{\dot\gamma}_k\,,\qquad [\mathcal{D}_a,\mathcal{D}_b]=-\boldsymbol{S}^2M_{ab}\,, \tag{2.3b}$$

with $\boldsymbol{S}^2:=\frac{1}{2}\boldsymbol{S}^{kl}\boldsymbol{S}_{kl}$. These anti-commutation relations follow from (A.9a)–(A.9c) by choosing the torsion to be covariantly constant.

[4] Compare with the case of 5D $\mathcal{N}=1$ anti-de Sitter superspace [26].

In accordance with the general supergravity definitions given in appendix A, the covariant derivatives include an appropriate SU(2) connection, see eq. (A.3). It follows from (2.3a), however, that the corresponding curvature is generated by a U(1) subgroup of SU(2). Therefore, one can gauge away most of the SU(2) connection except its U(1) part corresponding to the generator $\boldsymbol{S}^{kl} J_{kl}$

$$\Phi_A{}^{kl} J_{kl} \quad \longrightarrow \quad \Phi_A \, \boldsymbol{S}^{kl} J_{kl} \ . \tag{2.4}$$

In such a gauge, the torsion $\boldsymbol{S}^{ij}$ becomes constant,

$$\boldsymbol{S}^{ij} = \text{const} \ . \tag{2.5}$$

By applying a rigid SU(2) rotation to the covariant derivatives, we can always choose

$$\boldsymbol{S}^{\underline{12}} = 0 \ . \tag{2.6}$$

This choice will be often used in what follows.

2.1 $\mathcal{N} = 2$ AdS Killing supervectors: I

In this subsection, we do not assume any particular coordinate frame for the AdS covariant derivatives $\mathcal{D}_{\underline{A}}$. In particular, we do not impose the gauge fixing (2.4).

The isometry transformations of AdS$^{4|8}$ form the group OSp(2|4). Their explicit structure can be determined in a manner similar to the cases of 4D $\mathcal{N} = 1$ AdS superspace [7] and 5D $\mathcal{N} = 1$ superspace [26]. In the infinitesimal case, an isometry transformation is generated by a real supervector field $\xi^{\underline{A}} \mathcal{E}_{\underline{A}}$ such that the operator

$$\xi := \xi^{\underline{A}}(z) \mathcal{D}_{\underline{A}} = \xi^a \mathcal{D}_a + \xi^\alpha_i \mathcal{D}^i_\alpha + \bar{\xi}^i_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha}_i \tag{2.7}$$

enjoys the property

$$\Big[\xi + \frac{1}{2} \lambda^{cd} M_{cd} + \lambda^{kl} J_{kl}, \mathcal{D}_{\underline{A}} \Big] = 0 \, , \tag{2.8}$$

for some real antisymmetric tensor $\lambda^{cd}(z)$ and real symmetric tensor $\lambda^{kl}(z)$, $\overline{\lambda^{kl}} = \lambda_{kl}$. The latter equation implies

$$\big[\xi + \lambda^{kl} J_{kl}, \boldsymbol{S}^{ij} \big] = \big[\lambda^{kl} J_{kl}, \boldsymbol{S}^{ij} \big] = 0 \, , \tag{2.9}$$

and hence $\lambda^{kl} \propto \boldsymbol{S}^{kl}$. We therefore can replace (2.8) with

$$\Big[\xi + \frac{1}{2} \lambda^{cd} M_{cd} + \rho \boldsymbol{S}^{kl} J_{kl}, \mathcal{D}_{\underline{A}} \Big] = 0 \, , \tag{2.10}$$

for some real scalar $\rho(z)$. The meaning of eq. (2.10) is that the covariant derivatives do not change under the combined infinitesimal transformation consisting of coordinate (ξ), local Lorentz (λ^{cd}) and local U(1) (ρ) transformations. It turns out that eq. (2.10) uniquely determines the parameters λ^{cd} and ρ in terms of ξ. The $\xi^{\underline{A}} \mathcal{E}_{\underline{A}}$ is called a Killing

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supervector field. The set of all Killing supervector fields forms a Lie algebra, with respect to the standard Lie bracket, isomorphic to that of the group OSp(2|4).

Eq. (2.10) implies that the parameters $\xi^{\underline{A}}$, λ^{cd} and ρ are constrained as follows:

$$\begin{aligned} \mathcal{D}^i_\alpha \xi^\beta_j - \rho \boldsymbol{S}^i{}_j \delta^\beta_\alpha - \frac{1}{2}\lambda_\alpha{}^\beta \delta^i_j &= 0\,, && (2.11a)\\ \bar{\mathcal{D}}^{\dot\alpha}_i \xi^\beta_j - \frac{\mathrm{i}}{2}\boldsymbol{S}_{ij}\xi^{\dot\alpha\beta} &= 0\,, && (2.11b)\\ \bar{\mathcal{D}}^{\dot\alpha}_i \xi^b + 2\mathrm{i}\xi^\beta_i (\sigma^b)_\beta{}^{\dot\alpha} &= 0\,, && (2.11c)\\ \mathcal{D}^i_\alpha \lambda^{cd} - 4\boldsymbol{S}^{ij}\xi^\beta_j(\sigma^{cd})_{\alpha\beta} &= 0\,, && (2.11d)\\ \mathcal{D}^i_\alpha \rho - 2\xi^i_\alpha &= 0\,. && (2.11e) \end{aligned}$$

Note that eq. (2.11a) is equivalent to

$$\mathcal{D}^k_\gamma \xi^\gamma_k = \mathcal{D}^{(i}_{(\alpha}\xi^{j)}_{\beta)} = 0\,, \qquad 2\rho \boldsymbol{S}^{ij} + \mathcal{D}^{\gamma(i}\xi^{j)}_\gamma = 0\,, \qquad \lambda_{\alpha\beta} = \frac{1}{2}\mathcal{D}^k_{(\alpha}\xi_{\beta)k}\,. \tag{2.12}$$

Equation (2.11b) is equivalent to

$$\bar{\mathcal{D}}^{\dot\alpha}_k \xi^{\beta k} = 0\,, \qquad \bar{\mathcal{D}}^{\dot\alpha}_{(i}\xi^\beta_{j)} - \frac{\mathrm{i}}{2}\boldsymbol{S}_{ij}\xi^{\dot\alpha\beta} = 0\,. \tag{2.13}$$

Equation (2.11c) is equivalent to

$$\bar{\mathcal{D}}^{(\dot\alpha}_i \xi^{\dot\gamma)\gamma} = 0\,, \qquad \bar{\mathcal{D}}_{\dot\gamma i}\xi^{\dot\gamma\gamma} - 8\mathrm{i}\xi^\gamma_i = 0\,. \tag{2.14}$$

Equation (2.11d) is equivalent to

$$\bar{\mathcal{D}}^{\dot\alpha}_i \lambda^{\gamma\delta} = 0\,, \qquad \mathcal{D}^i_{(\alpha}\lambda_{\gamma\delta)} = 0\,, \qquad \mathcal{D}^{\gamma i}\lambda_{\gamma\delta} + 6\boldsymbol{S}^{ij}\xi_{\delta j} = 0\,. \tag{2.15}$$

It is also worth noting that the above equations imply

$$\mathcal{D}_{(a}\xi_{b)} = 0 \tag{2.16}$$

which is a natural generalization of the standard equation for Killing vectors.

Similar to the case of 5D $\mathcal{N}=1$ AdS superspace [26], all the components $\xi^{\underline{A}}$ can be expressed in terms of the scalar parameter ρ as follows:

$$\xi^\alpha_i = \frac{1}{2}\mathcal{D}^\alpha_i \rho\,, \qquad \xi_{\alpha\dot\beta} = \frac{\mathrm{i}}{2\boldsymbol{S}^2}\boldsymbol{S}_{ij}\mathcal{D}^i_\alpha \bar{\mathcal{D}}^j_{\dot\beta}\rho\,, \qquad \lambda_{\alpha\beta} = \frac{1}{4}\mathcal{D}^k_\alpha \mathcal{D}_{\beta k}\rho\,. \tag{2.17}$$

The latter obeys a number of constraints including

$$\Big(\mathcal{D}^{\gamma i}\mathcal{D}^j_\gamma + 4\boldsymbol{S}^{ij}\Big)\rho = 0\,, \qquad \Big(\mathcal{D}^i_\alpha \bar{\mathcal{D}}^j_{\dot\beta} - \frac{1}{2\boldsymbol{S}^2}\boldsymbol{S}^{ij}\boldsymbol{S}_{kl}\mathcal{D}^k_\alpha \bar{\mathcal{D}}^l_{\dot\beta}\Big)\rho = 0\,, \tag{2.18}$$

and hence

$$\mathcal{D}_a \rho = 0\,. \tag{2.19}$$

2.2 $\mathcal{N}=1$ reduction

It is of interest to work out $\mathcal{N}=1$ components of the $\mathcal{N}=2$ Killing supervectors, as well as of covariant $\mathcal{N}=2$ supermultiplets. Given a tensor superfied $U(x,\theta_i,\bar\theta^i)$ in $\mathcal{N}=2$ AdS superspace, we introduce its $\mathcal{N}=1$ projection

$$U| := U(x,\theta_i,\bar\theta^i)|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0} \tag{2.20}$$

in a *special coordinate system* to be specified below. For the covariant derivatives

$$\mathcal{D}_{\underline{A}} = \mathcal{E}_{\underline{A}}{}^{\underline{M}}\partial_{\underline{M}} + \frac{1}{2}\Omega_{\underline{A}}{}^{bc}M_{bc} + \Phi_{\underline{A}}\boldsymbol{S}^{kl}J_{kl}\,, \tag{2.21}$$

the projection is defined according to

$$\mathcal{D}_{\underline{A}}| := \mathcal{E}_{\underline{A}}{}^{\underline{M}}|\partial_{\underline{M}}| + \frac{1}{2}\Omega_{\underline{A}}{}^{bc}|M_{bc} + \Phi_{\underline{A}}|\boldsymbol{S}^{kl}J_{kl}\ . \tag{2.22}$$

Here the first term on the right, $\mathcal{E}_{\underline{A}}{}^{\underline{M}}|\partial_{\underline{M}}|$, includes the partial derivatives with respect to the local coordinates of $\mathcal{N}=2$ AdS superspace.

With the choice $\boldsymbol{S}^{\underline{12}}=0$, as in eq. (2.6), it follows from (2.3a) and (2.3b) that

$$\{\mathcal{D}^{\underline{1}}_\alpha,\mathcal{D}^{\underline{1}}_\beta\} = 4\boldsymbol{S}^{\underline{11}}M_{\alpha\beta}\,,\quad \{\mathcal{D}^{\underline{1}}_\alpha,\bar{\mathcal{D}}^{\dot\beta}_{\underline{1}}\} = -2{\rm i}(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c\,,\quad [\mathcal{D}_a,\mathcal{D}^{\underline{1}}_\beta] = \frac{\rm i}{2}(\sigma_a)_{\beta\dot\gamma}\boldsymbol{S}^{\underline{11}}\bar{\mathcal{D}}^{\dot\gamma}_{\underline{1}}\ . \tag{2.23}$$

Therefore, the operators $(\mathcal{D}_a,\,\mathcal{D}^{\underline{1}}_\alpha,\,\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}})$ form a closed algebra which is in fact isomorphic to that of the covariant derivatives for $\mathcal{N}=1$ AdS superspace with

$$\bar\mu = -\boldsymbol{S}^{\underline{11}}\,, \tag{2.24}$$

see appendix C. Note also that no U(1) curvature is present in (2.23).

We use the freedom to perform general coordinate, local Lorentz and U(1) transformations to choose the gauge

$$\mathcal{D}^{\underline{1}}_\alpha| = \nabla_\alpha\,,\qquad \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}| = \bar\nabla^{\dot\alpha}\,, \tag{2.25}$$

with $\nabla_A=(\nabla_a,\nabla_\alpha,\bar\nabla^{\dot\alpha})$ the covariant derivatives for $\mathcal{N}=1$ anti-de Sitter superspace (see appendix C). In such a coordinate system, the operators $\mathcal{D}^{\underline{1}}_\alpha|$ and $\bar{\mathcal{D}}_{\dot\alpha\underline{1}}|$ do not involve any partial derivatives with respect to $\theta_{\underline{2}}$ and $\bar\theta^{\underline{2}}$, and therefore, for any positive integer k, it holds that $\big(\mathcal{D}_{\hat\alpha_1}\cdots\mathcal{D}_{\hat\alpha_k}U\big)\big| = \mathcal{D}_{\hat\alpha_1}|\cdots\mathcal{D}_{\hat\alpha_k}|U|$, where $\mathcal{D}_{\hat\alpha} := (\mathcal{D}^{\underline{1}}_\alpha,\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}})$ and U is a tensor superfield.

Given an arbitrary $\mathcal{N}=2$ AdS Killing supervector ξ, we consider its $\mathcal{N}=1$ projection

$$\xi| = \lambda^a\nabla_a + \lambda^\alpha\nabla_\alpha + \bar\lambda_{\dot\alpha}\bar\nabla^{\dot\alpha} + \varepsilon^\alpha\mathcal{D}^{\underline{2}}_\alpha| + \bar\varepsilon_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}}|\,, \tag{2.26}$$

where we have defined

$$\lambda^a := \xi^a|\,,\quad \lambda^\alpha := \xi^\alpha_{\underline{1}}|\,,\quad \bar\lambda_{\dot\alpha} := \bar\xi^{\underline{1}}_{\dot\alpha}|\,,\qquad \varepsilon^\alpha := \xi^\alpha_{\underline{2}}|\,,\quad \bar\varepsilon_{\dot\alpha} := \xi^{\underline{2}}_{\dot\alpha}|\ . \tag{2.27}$$

We also introduce the projections of the parameters λ_{ab} and ρ:

$$\omega_{ab} := \lambda_{ab}|\,,\qquad \varepsilon := \rho|\ . \tag{2.28}$$

Now, the OSp(2|4) transformation law of a tensor superfield U,

$$\delta U = \Big(\xi + \frac{1}{2}\lambda^{cd}M_{cd} + \rho \boldsymbol{S}^{kl}J_{kl}\Big)U\,, \tag{2.29}$$

turns into

$$\begin{aligned}\delta U| = &\Big(\lambda^a\nabla_a + \lambda^\alpha\nabla_\alpha + \bar\lambda_{\dot\alpha}\bar\nabla^{\dot\alpha} + \frac{1}{2}\omega^{ab}M_{ab}\Big)U| \\ &+\Big(\varepsilon^\alpha(\mathcal{D}^{\underline{2}}_\alpha U)| + \bar\varepsilon_{\dot\alpha}(\bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}}U)|\Big) \;-\; \varepsilon(\bar\mu J_{\underline{11}} + \mu J_{\underline{22}})U|\,,\end{aligned} \tag{2.30}$$

where we have made use of (2.24). It can be shown that $\Lambda = \lambda^a\nabla_a + \lambda^\alpha\nabla_\alpha + \bar\lambda_{\dot\alpha}\bar\nabla^{\dot\alpha}$ is an $\mathcal{N}=1$ AdS Killing supervector (see appendix C), and the variation in the first line of (2.30) is the infinitesimal OSp(1|4) transformation generated by Λ. The parameters ε^α, $\bar\varepsilon_{\dot\alpha}$ and ε generate the second supersymmetry and U(1) transformations. It can be shown, using eqs. (2.12)–(2.15), that they obey the constraints [21]

$$\varepsilon^\alpha = \frac{1}{2}\nabla^\alpha\varepsilon\,, \qquad \nabla_\alpha\bar\nabla^{\dot\alpha}\varepsilon = 0\,, \qquad \big(\nabla^2 - 4\bar\mu\big)\varepsilon = 0\;. \tag{2.31}$$

2.3 Projective supermultiplets in AdS$^{4|8}$

General matter couplings in 4D $\mathcal{N}=2$ supergravity can be described in terms of covariant projective supermultiplets [1]. Here we briefly introduce such multiplets in the case of $\mathcal{N}=2$ AdS superspace, and then work out their reduction to $\mathcal{N}=1$ superfields.

In the superspace AdS$^{4|8}$, a projective supermultiplet of weight n, $\mathcal{Q}^{(n)}(z,u^+)$, is defined to be a scalar superfield that lives on AdS$^{4|8}$, is holomorphic with respect to the isotwistor variables u^+_i on an open domain of $\mathbb{C}^2\setminus\{0\}$, and is characterized by the following conditions:

(1) it obeys the covariant analyticity constraints[5]

$$\mathcal{D}^+_\alpha\mathcal{Q}^{(n)} = \bar{\mathcal{D}}^+_{\dot\alpha}\mathcal{Q}^{(n)} = 0\,, \qquad \mathcal{D}^+_\alpha := u^+_i\mathcal{D}^i_\alpha\,, \quad \bar{\mathcal{D}}^+_{\dot\alpha} := u^+_i\bar{\mathcal{D}}^i_{\dot\alpha}\;; \tag{2.32}$$

(2) it is a homogeneous function of u^+ of degree n, that is,

$$\mathcal{Q}^{(n)}(z,c\,u^+) \;=\; c^n\,\mathcal{Q}^{(n)}(z,u^+)\,, \qquad c\in\mathbb{C}\setminus\{0\}\;; \tag{2.33}$$

(3) the infinitesimal OSp(2|4) transformations act on $\mathcal{Q}^{(n)}$ as follows:

$$\begin{aligned}\delta_\xi\mathcal{Q}^{(n)} \;&=\; \Big(\xi^a\mathcal{D}_a + \xi^\alpha_i\mathcal{D}^i_\alpha + \bar\xi^i_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}_i + \rho\boldsymbol{S}^{ij}J_{ij}\Big)\mathcal{Q}^{(n)}\,,\\ \boldsymbol{S}^{ij}J_{ij}\mathcal{Q}^{(n)} \;&=\; -\frac{1}{(u^+u^-)}\Big(\boldsymbol{S}^{++}D^{--} - n\,\boldsymbol{S}^{+-}\Big)\mathcal{Q}^{(n)}\,, \qquad \boldsymbol{S}^{\pm\pm} = \boldsymbol{S}^{ij}\,u^\pm_i u^\pm_j\,,\end{aligned} \tag{2.34}$$

with $D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}$. The transformation law (2.34) involves an additional two-vector, u^-_i, which is only subject to the condition $(u^+u^-) := u^{+i}u^-_i \neq 0$, and is otherwise completely arbitrary. Both $\mathcal{Q}^{(n)}$ and $\boldsymbol{S}^{ij}J_{ij}\mathcal{Q}^{(n)}$ are independent of u^-.

[5] In the rigid supersymmetric case, constraints of the form (2.32) in isotwistor superspace $\mathbb{R}^{4|8}\times\mathbb{C}P^1$ were first introduced by Rosly [27], and later by the harmonic [24] and projective [11, 12] superspace practitioners.

In the family of projective multiplets, one can introduce a generalized conjugation, $\mathcal{Q}^{(n)} \to \widetilde{\mathcal{Q}}^{(n)}$, defined as

$$\widetilde{\mathcal{Q}}^{(n)}(u^+) \equiv \bar{\mathcal{Q}}^{(n)}\big(\overline{u^+} \to \widetilde{u}^+\big)\,, \qquad \widetilde{u}^+ = \mathrm{i}\,\sigma_2\, u^+\,, \tag{2.35}$$

with $\bar{\mathcal{Q}}^{(n)}(\overline{u^+})$ the complex conjugate of $\mathcal{Q}^{(n)}(u^+)$. It is easy to check that $\widetilde{\mathcal{Q}}^{(n)}(z,u^+)$ is a projective multiplet of weight n. One can also see that $\widetilde{\widetilde{\mathcal{Q}}}^{(n)} = (-1)^n\,\mathcal{Q}^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. The $\widetilde{\mathcal{Q}}^{(n)}$ is called the smile-conjugate of $\mathcal{Q}^{(n)}$.

It is natural to interpret the variables u^+_i as homogeneous coordinates for $\mathbb{C}P^1$. Due to the homogeneity condition (2.33), the projective multiplets $\mathcal{Q}^{(n)}(z,u^+)$ actually depend on a single complex variable ζ which is an inhomogeneous local complex coordinate for $\mathbb{C}P^1$. To describe the projective multiplets in terms of ζ, one should replace $\mathcal{Q}^{(n)}(z,u^+)$ with a new superfield $\mathcal{Q}^{[n]}(z,\zeta) \propto \mathcal{Q}^{(n)}(z,u^+)$, where $\mathcal{Q}^{[n]}(z,\zeta)$ is holomorphic with respect to ζ. The explicit definition of $\mathcal{Q}^{[n]}(\zeta)$ depends on the supermultiplet under consideration. One can cover $\mathbb{C}P^1$ by two open charts in which ζ can be defined, and the simplest choice is:

(i) the north chart characterized by $u^{+\underline{1}} \neq 0$;

(ii) the south chart with $u^{+\underline{2}} \neq 0$.

Our consideration will be restricted to the north chart in which the variable $\zeta \in \mathbb{C}$ is defined as

$$u^{+i} = u^{+\underline{1}}(1,\zeta) = u^{+\underline{1}}\zeta^i\,, \qquad \zeta^i = (1,\zeta)\,, \qquad \zeta_i = \varepsilon_{ij}\,\zeta^j = (-\zeta,1)\;. \tag{2.36}$$

In this chart, we can choose

$$u^-_i = (1,0)\,, \qquad u^{-i} = \varepsilon^{ij}\,u^-_j = (0,-1)\;. \tag{2.37}$$

Before discussing the possible types of $\mathcal{Q}^{[n]}(\zeta)$, let us first turn to the $\mathrm{U}(1)$ part of the transformation law (2.34). The parameters $\boldsymbol{S}^{++}$ and $\boldsymbol{S}^{+-}$ in (2.34) can be represented as $\boldsymbol{S}^{++} = \big(u^{+\underline{1}}\big)^2\Xi(\zeta)$ and $\boldsymbol{S}^{+-} = u^{+\underline{1}}\Delta(\zeta)$, where

$$\Xi(\zeta) = \boldsymbol{S}^{ij}\,\zeta_i\zeta_j = \boldsymbol{S}^{\underline{11}}\,\zeta^2 - 2\boldsymbol{S}^{\underline{12}}\,\zeta + \boldsymbol{S}^{\underline{22}}\,, \qquad \Delta(\zeta) = \boldsymbol{S}^{\underline{1}i}\,\zeta_i = -\boldsymbol{S}^{\underline{11}}\,\zeta + \boldsymbol{S}^{\underline{12}}\;. \tag{2.38}$$

Now, let us introduce the major projective supermultiplet $\mathcal{Q}^{(n)}(z,u^+)$ and the corresponding superfields $\mathcal{Q}^{[n]}(z,\zeta)$. In the case of covariant arctic weight-n hypermultiplets $\Upsilon^{(n)}(z,u^+)$ [1], it is natural to define

$$\Upsilon^{(n)}(z,u^+) = (u^{+\underline{1}})^n\Upsilon^{[n]}(z,\zeta)\,, \qquad \Upsilon^{[n]}(z,\zeta) = \sum_{k=0}^{\infty}\Upsilon_k(z)\zeta^k\;. \tag{2.39}$$

The corresponding $\mathrm{U}(1)$ transformation law is:

$$\rho\boldsymbol{S}^{ij}J_{ij}\Upsilon^{[n]}(\zeta) = \rho\Big(\Xi(\zeta)\,\partial_\zeta + n\,\Delta(\zeta)\Big)\Upsilon^{[n]}(\zeta)\;. \tag{2.40}$$

The smile-conjugate of $\Upsilon^{(n)}$ is called a covariant *antarctic* weight-n multiplet. In this case

$$\widetilde{\Upsilon}^{(n)}(z,u) \,=\, (u^{+\underline{2}})^n\, \widetilde{\Upsilon}^{[n]}(z,\zeta)\,, \qquad \widetilde{\Upsilon}^{[n]}(z,\zeta) = \sum_{k=0}^{\infty}(-1)^k \bar{\Upsilon}_k(z)\frac{1}{\zeta^k}\,, \tag{2.41}$$

with $\bar{\Upsilon}_k$ the complex conjugate of Υ_k. The U(1) transformation of $\widetilde{\Upsilon}^{[n]}(z,\zeta)$ is as follows:

$$\rho \boldsymbol{S}^{ij} J_{ij} \widetilde{\Upsilon}^{[n]}(\zeta) = \frac{\rho}{\zeta^n}\Big(\Xi(\zeta)\,\partial_\zeta + n\,\Delta(\zeta)\Big)\Big(\zeta^n\,\widetilde{\Upsilon}^{(n)}(\zeta)\Big)\;. \tag{2.42}$$

In the case of a real weight-$2n$ projective superfield $R^{(2n)}(z,u^+)$, it is natural to define

$$R^{(2n)}(z,u^+) \,=\, \big(\mathrm{i}\, u^{+\underline{1}}u^{+\underline{2}}\big)^n R^{[2n]}(z,\zeta)\;. \tag{2.43}$$

The U(1) transformation of $R^{[2n]}(z,\zeta)$ is:

$$\rho \boldsymbol{S}^{ij} J_{ij} R^{[2n]} = \frac{\rho}{\zeta^n}\Big(\Xi(\zeta)\,\partial_\zeta + 2n\,\Delta(\zeta)\Big)\Big(\zeta^n R^{[2n]}\Big)\;. \tag{2.44}$$

There are two major types of superfields $R^{[2n]}(z,\zeta)$: a real $O(2n)$-multiplet $(n = 1, 2, \dots)$

$$H^{[2n]}(z,\zeta) \,=\, \sum_{k=-n}^{n} H_k(z)\zeta^k\,, \qquad \bar{H}_k = (-1)^k H_{-k}\,, \tag{2.45}$$

and a tropical weight-$2n$ multiplet

$$U^{[2n]}(z,\zeta) \,=\, \sum_{k=-\infty}^{\infty} U_k(z)\zeta^k\,, \qquad \bar{U}_k = (-1)^k U_{-k}\;. \tag{2.46}$$

If the projective supermultiplet $\mathcal{Q}^{(n)}(z,u^+)$ is described by $\mathcal{Q}^{[n]}(z,\zeta) \propto \mathcal{Q}^{(n)}(z,u^+)$, then the covariant analyticity conditions (2.32) become

$$\mathcal{D}^+_\alpha(\zeta)\,\mathcal{Q}^{[n]}(\zeta) = 0\,, \qquad \mathcal{D}^+_\alpha(\zeta) = -\mathcal{D}^i_\alpha\zeta_i = \zeta\,\mathcal{D}^{\underline{1}}_\alpha - \mathcal{D}^{\underline{2}}_\alpha\,, \tag{2.47a}$$

$$\bar{\mathcal{D}}^{+\dot{\alpha}}(\zeta)\,\mathcal{Q}^{[n]}(\zeta) = 0\,, \qquad \bar{\mathcal{D}}^{+\dot{\alpha}}(\zeta) = \bar{\mathcal{D}}^{\dot{\alpha}}_i\zeta^i = \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} + \zeta\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}}\,, \tag{2.47b}$$

and therefore

$$\mathcal{D}^{\underline{2}}_\alpha\mathcal{Q}^{[n]}(\zeta) = \zeta\,\mathcal{D}^{\underline{1}}_\alpha\mathcal{Q}^{[n]}(\zeta)\,, \quad \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}}\,\mathcal{Q}^{[n]}(\zeta) = -\frac{1}{\zeta}\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}}\mathcal{Q}^{[n]}(\zeta)\;. \tag{2.48}$$

The differential operator $\xi^\alpha_i\mathcal{D}^i_\alpha + \bar{\xi}^i_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}_i$, which enters the transformation law (2.34), acts on $\mathcal{Q}^{[n]}(\zeta)$ as

$$\big(\xi^\alpha_i\mathcal{D}^i_\alpha + \bar{\xi}^i_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}_i\big)\,\mathcal{Q}^{[n]}(\zeta) = \Big((\xi^\alpha_{\underline{1}} + \zeta\xi^\alpha_{\underline{2}})\mathcal{D}^{\underline{1}}_\alpha + \big(\bar{\xi}^{\underline{1}}_{\dot{\alpha}} - \frac{1}{\zeta}\bar{\xi}^{\underline{2}}_{\dot{\alpha}}\big)\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}}\Big)\,\mathcal{Q}^{[n]}(\zeta)\;. \tag{2.49}$$

Let us impose the SU(2) gauge (2.4) and choose $\boldsymbol{S}^{\underline{12}} = 0$, as in eq. (2.6). Then, eq. (2.49) implies that the $\mathcal{N} = 1$ projection of $\xi\,\mathcal{Q}^{[n]}(\zeta)$ is

$$\Big(\xi\,\mathcal{Q}^{[n]}(\zeta)\Big)\Big| = \Lambda\,\mathcal{Q}^{[n]}(\zeta)\big| + \left(\zeta\varepsilon^\alpha\nabla_\alpha - \frac{1}{\zeta}\bar{\varepsilon}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}\right)\mathcal{Q}^{[n]}(\zeta)\big|\,, \tag{2.50}$$

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with ξ an arbitrary $\mathcal{N}=2$ AdS Killing supervector, and Λ the induced $\mathcal{N}=1$ AdS Killing supervector. As a result, the $\mathcal{N}=1$ projection of the transformation $\delta_\xi \Upsilon^{[n]}(\zeta)$ becomes

$$\begin{aligned}\delta_\xi \Upsilon^{[n]}(\zeta)\big| &= \Lambda \Upsilon^{[n]}(\zeta)\big| \\ &\quad+\Big(\zeta\varepsilon^\alpha\nabla_\alpha - \frac{1}{\zeta}\bar\varepsilon_{\dot\alpha}\bar\nabla^{\dot\alpha}\Big)\Upsilon^{[n]}(\zeta)\big| + \varepsilon\big(\Xi(\zeta)\,\partial_\zeta + n\,\Delta(\zeta)\big)\Upsilon^{[n]}(\zeta)\big|\,, \end{aligned}\tag{2.51}$$

and similarly for $\delta_\xi \widetilde{\Upsilon}^{[n]}(\zeta)\big|$. The $\mathcal{N}=1$ projection of the transformation $\delta_\xi R^{[2n]}(\zeta)$ becomes

$$\begin{aligned}\delta_\xi R^{[2n]}(\zeta)\big| &= \Lambda R^{[2n]}(\zeta)\big| \\ &\quad+\Big(\zeta\varepsilon^\alpha\nabla_\alpha - \frac{1}{\zeta}\bar\varepsilon_{\dot\alpha}\bar\nabla^{\dot\alpha}\Big)R^{[2n]}(\zeta)\big| + \frac{\varepsilon}{\zeta^n}\Big(\Xi(\zeta)\,\partial_\zeta + 2n\,\Delta(\zeta)\Big)\big(\zeta^n R^{[2n]}\big|\big)\,. \end{aligned}\tag{2.52}$$

In the gauge chosen, the parameters $\Xi(\zeta)$ and $\Delta(\zeta)$ are:

$$\Xi(\zeta) = -\bar\mu\,\zeta^2 - \mu\,, \qquad \Delta(\zeta) = \bar\mu\,\zeta\,. \tag{2.53}$$

3. Dynamics in AdS$^{4|8}$

In the case of $\mathcal{N}=2$ anti-de Sitter space, the action (1.1) becomes

$$S = \frac{1}{2\pi}\oint (u^+ \mathrm{d}u^+)\int \mathrm{d}^4x\,\mathrm{d}^4\theta \mathrm{d}^4\bar\theta\,\mathcal{E}\,\frac{\mathcal{L}^{++}}{(\boldsymbol{S}^{++})^2}\,, \tag{3.1}$$

where the Lagrangian $\mathcal{L}^{++}(z,u^+)$ is a real weight-two projective supermultiplet.

It is worth giving two non-trivial examples of supersymmetric theories in AdS$^{4|8}$. First, we consider a superconformal model of arctic weight-one hypermultiplets Υ^+ and their smile-conjugates $\widetilde{\Upsilon}^+$ described by the Lagrangian [28, 29]

$$\mathcal{L}^{++}_{\rm conf} = \mathrm{i}\,K(\Upsilon^+, \widetilde{\Upsilon}^+)\,, \tag{3.2}$$

where the real function $K(\Phi^I, \bar\Phi^{\bar J})$ obeys the homogeneity condition

$$\Phi^I\frac{\partial}{\partial\Phi^I}K(\Phi,\bar\Phi) = K(\Phi,\bar\Phi)\,. \tag{3.3}$$

Our second example is the non-superconformal model of arctic weight-zero multiplets $\boldsymbol{\Upsilon}$ and their smile-conjugates $\widetilde{\boldsymbol{\Upsilon}}$ described by the Lagrangian [26]

$$\mathcal{L}^{++}_{\rm non-conf} = \boldsymbol{S}^{++}\,\mathbf{K}(\boldsymbol{\Upsilon}, \widetilde{\boldsymbol{\Upsilon}})\,, \tag{3.4}$$

with $\mathbf{K}(\Phi^I, \bar\Phi^{\bar J})$ a real function which is not required to obey any homogeneity condition. The action is invariant under Kähler transformations of the form

$$\mathbf{K}(\boldsymbol{\Upsilon}, \widetilde{\boldsymbol{\Upsilon}}) \;\to\; \mathbf{K}(\boldsymbol{\Upsilon}, \widetilde{\boldsymbol{\Upsilon}}) + \boldsymbol{\Lambda}(\boldsymbol{\Upsilon}) + \bar{\boldsymbol{\Lambda}}(\widetilde{\boldsymbol{\Upsilon}})\,, \tag{3.5}$$

with $\boldsymbol{\Lambda}(\Phi^I)$ a holomorphic function.

Throughout this section, the torsion $\boldsymbol{S}^{ij}$ is chosen to obey eq. (2.6).

3.1 Projecting the $\mathcal{N}=2$ action into $\mathcal{N}=1$ superspace: I

In this subsection, we reduce the $\mathcal{N}=2$ supersymmetric action (3.1) to the $\mathcal{N}=1$ AdS superspace.

Without loss of generality, the integration contour in (3.1) can be assumed to lie outside the north pole $u^{+i} \propto (0,1)$, and then we can use the complex variable ζ defined in the north chart, eq. (2.36), to parametrize the projective supermultiplets. Associated with the Lagrangian $\mathcal{L}^{++}(u^+)$ is the superfield $\mathcal{L}(\zeta)$ defined as

$$\mathcal{L}^{++}(u^+) := \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\mathcal{L}(\zeta) = \mathrm{i}(u^{+\underline{1}})^2\zeta\,\mathcal{L}(\zeta)\ . \tag{3.6}$$

Similarly, associated with $\boldsymbol{S}^{++}(u^+)$ is the superfield $\boldsymbol{S}(\zeta)$ defined as

$$\boldsymbol{S}^{++}(u^+) := \mathrm{i}(u^{+\underline{1}})^2\zeta\,\boldsymbol{S}(\zeta)\,, \qquad \boldsymbol{S}(\zeta) = \mathrm{i}\left(\bar{\mu}\,\zeta + \mu\,\frac{1}{\zeta}\right)\ . \tag{3.7}$$

Let $\mathcal{L}(\zeta)|$ denote the $\mathcal{N}=1$ projection of the Lagrangian $\mathcal{L}(\zeta)$. Then, the manifestly $\mathcal{N}=2$ supersymmetric functional (3.1) can be shown to be equivalent to the following action in $\mathrm{AdS}^{4|4}$:

$$S = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\mathcal{L}(\zeta)|\,, \qquad E^{-1} := \mathrm{Ber}(E_A{}^M)\ . \tag{3.8}$$

While this form of the action will be derived in section 5, here we only demonstrate that (3.8) is invariant under the $\mathrm{OSp}(2|4)$ transformations. We note that the transformation law of $\mathcal{L}(\zeta)$ is given by eq. (2.52) with $n=1$. It is obvious that (3.8) is manifestly invariant under the $\mathcal{N}=1$ AdS transformations

$$\delta_\Lambda\mathcal{L}(\zeta)| = \Lambda\mathcal{L}(\zeta)| = \left(\lambda^a\nabla_a + \lambda^\alpha\nabla_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}\right)\mathcal{L}(\zeta)|\ . \tag{3.9}$$

The other transformations, which are generated by the parameters ε, ε^α, $\bar{\varepsilon}_{\dot{\alpha}}$ in (2.52), act on $\mathcal{L}(\zeta)|$ as follows:

$$\delta_\varepsilon\mathcal{L}(\zeta)| = \left(\zeta\varepsilon^\alpha\nabla_\alpha - \frac{1}{\zeta}\bar{\varepsilon}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}\right)\mathcal{L}(\zeta)| - \frac{\varepsilon}{\zeta}\Big((\zeta^2\bar{\mu}+\mu)\partial_\zeta - 2\zeta\bar{\mu}\Big)\big(\zeta\mathcal{L}(\zeta)|\big)\ . \tag{3.10}$$

The corresponding variation of the action,

$$\delta_\varepsilon S = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\delta_\varepsilon\mathcal{L}(\zeta)|\,, \tag{3.11}$$

can be transformed by integrating by parts the derivatives ∇_α, $\bar{\nabla}^{\dot{\alpha}}$ and ∂_ζ. This leads to

$$\delta_\varepsilon S = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\Big(-\zeta(\nabla^\alpha\varepsilon_\alpha) + \frac{1}{\zeta}(\bar{\nabla}_{\dot{\alpha}}\bar{\varepsilon}^{\dot{\alpha}}) + 2\varepsilon\Big(\bar{\mu}\zeta - \mu\frac{1}{\zeta}\Big)\Big)\mathcal{L}(\zeta)| = 0\,,$$

where we have made use of the relations

$$\varepsilon^\alpha = \frac{1}{2}\nabla^\alpha\varepsilon\,, \qquad \nabla^\alpha\varepsilon_\alpha = 2\bar{\mu}\varepsilon\ . \tag{3.12}$$

3.2 Free hypermultiplets, dual tensor multiplets and some generalizations

To get a better feeling of the sigma-models (3.3) and (3.4), it is instructive to examine their simplest versions corresponding to free hypermultiplets.

Consider the Lagrangian

$$\mathcal{L}^{++}_{\rm conf} = {\rm i}\,\widetilde{\Upsilon}^{+}\Upsilon^{+} \tag{3.13}$$

which describes the dynamics of a weight-one arctic hypermultiplet Υ^{+} and its smile-conjugate $\widetilde{\Upsilon}^{+}$.

We represent $\Upsilon^{+}(u^{+}) = u^{+\underline{1}}\Upsilon(\zeta)$, where $\Upsilon(\zeta)$ is given by a convergent Taylor series centered at $\zeta = 0$. Then, the analyticity conditions (2.48) imply

$$\Upsilon(\zeta)| = \Phi + \zeta\Gamma + \sum_{k=2}^{+\infty}\zeta^{k}\Upsilon_{k}|\,, \qquad \bar{\nabla}^{\dot{\alpha}}\Phi = 0\,, \qquad \big(\bar{\nabla}^{2} - 4\mu\big)\,\Gamma = 0\;. \tag{3.14}$$

Here Φ and Γ are covariantly chiral and complex linear superfields, respectively, while the higher-order components $\Upsilon_{k}|$, with $k = 2, 3, \dots$, are complex unconstrained superfields. It is useful to recall that, in the $\mathcal{N} = 1$ AdS superspace, the chirality constraint $\bar{\nabla}^{\dot{\alpha}}\Phi = 0$ is equivalent to $\bar{\nabla}^{2}\Phi = 0$ [18]. Moreover, any complex scalar superfield U can be uniquely represented in the form $U = \Phi + \Gamma$, for some chiral Φ and complex linear Γ scalars [18] (see [51] for a nice review of the $\mathcal{N} = 1$ AdS supermultiplets classified in [18]).

Then, evaluating the action (3.8) with $\mathcal{L}(\zeta)$ corresponding to (3.13) gives

$$\begin{aligned} S_{\rm conf} &= \oint_{C}\frac{{\rm d}\zeta}{2\pi{\rm i}\zeta}\int {\rm d}^{4}x\,{\rm d}^{2}\theta{\rm d}^{2}\bar{\theta}\,E\,\widetilde{\Upsilon}(\zeta)|\Upsilon(\zeta)| \\ &= \int {\rm d}^{4}x\,{\rm d}^{2}\theta{\rm d}^{2}\bar{\theta}\,E\left(\bar{\Phi}\Phi - \bar{\Gamma}\Gamma + \sum_{k=2}^{+\infty}(-1)^{k}\bar{\Upsilon}_{k}|\Upsilon_{k}|\right)\;. \end{aligned} \tag{3.15}$$

Integrating out the auxiliary superfields $\Upsilon_{k}|$, in complete analogy with the flat case [30], reduces the action to

$$S_{\rm conf} = \int {\rm d}^{4}x\,{\rm d}^{2}\theta{\rm d}^{2}\bar{\theta}\,E\,\big(\bar{\Phi}\Phi - \bar{\Gamma}\Gamma\big)\;. \tag{3.16}$$

The first term in the action provides the standard (or *minimal*) off-shell description of $\mathcal{N} = 1$ massless scalar multiplet. The second term describes the same multiplet on the mass shell, although it is realized in terms of a complex scalar and its conjugate. The latter description is known as the *non-minimal* scalar multiplet [31].

The action (3.16) is manifestly $\mathcal{N} = 1$ supersymmetric. It is also invariant under the second SUSY and U(1) transformations which are generated by a real parameter ε subject to the constraints (2.31), and have the form:

$$\delta_{\varepsilon}\Phi = -\big(\bar{\varepsilon}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}} + \varepsilon\mu\big)\,\Gamma = -\frac{1}{4}(\bar{\nabla}^{2} - 4\mu)(\varepsilon\Gamma)\,, \qquad \delta_{\varepsilon}\Gamma = (\varepsilon^{\alpha}\nabla_{\alpha} + \varepsilon\bar{\mu})\,\Phi\;. \tag{3.17}$$

The complex linear superfield Γ can be dualized[6] into a covariantly chiral scalar superfield Ψ, $\bar{\nabla}^{\dot{\alpha}}\Psi = 0$, by applying a superfield Legendre transformation [32] (see [6, 7] for reviews) to end up with

$$S^{(\mathrm{dual})}_{\mathrm{conf}} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\left(\bar{\Phi}\Phi + \bar{\Psi}\Psi\right) \ . \tag{3.18}$$

The second SUSY and $\mathrm{U}(1)$ invariance of this model is as follows:

$$\delta_\varepsilon \Phi = -\frac{1}{4}(\bar{\nabla}^2 - 4\mu)(\varepsilon\bar{\Psi})\,, \qquad \delta_\varepsilon \Psi = \frac{1}{4}(\bar{\nabla}^2 - 4\mu)(\varepsilon\bar{\Phi}) \ . \tag{3.19}$$

Now consider the Lagrangian

$$\mathcal{L}^{++}_{\mathrm{non-conf}} = \frac{\boldsymbol{S}^{++}}{|\boldsymbol{S}|}\widetilde{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon} \tag{3.20}$$

describing the dynamics of a weight-zero arctic multiplet $\boldsymbol{\Upsilon}$ and its conjugate $\widetilde{\boldsymbol{\Upsilon}}$. Upon reduction to the $\mathcal{N}=1$ AdS superspace, this system is described by the action

$$S_{\mathrm{non-conf}} = \frac{1}{|\mu|}\oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \boldsymbol{S}(\zeta)\widetilde{\boldsymbol{\Upsilon}}(\zeta)|\boldsymbol{\Upsilon}(\zeta)|\,, \tag{3.21}$$

where $\boldsymbol{S}(\zeta)$ is given in eq. (3.7). The $\mathcal{N}=1$ projection of $\boldsymbol{\Upsilon}(\zeta)$ has the form:

$$\boldsymbol{\Upsilon}|(\zeta) = \boldsymbol{\Phi} + \zeta\boldsymbol{\Gamma} + \sum_{k=2}^{+\infty}\zeta^k\boldsymbol{\Upsilon}_k|\,, \qquad \bar{\nabla}^{\dot{\alpha}}\boldsymbol{\Phi} = 0\,, \qquad \left(\bar{\nabla}^2 - 4\mu\right)\boldsymbol{\Gamma} = 0\,, \tag{3.22}$$

with the scalar superfields $\boldsymbol{\Upsilon}_k|$, $k \geq 2$, being complex unconstrained. To perform the contour integral in (3.21), it is useful to note that

$$\frac{1}{|\mu|}\boldsymbol{S}(\zeta) = \Big(1 - \frac{1}{\lambda}\frac{1}{\zeta}\Big)\Big(1 + \lambda\zeta\Big)\,, \qquad \lambda := \mathrm{i}\,\frac{\bar{\mu}}{|\mu|} \ . \tag{3.23}$$

We then can redefine the components of the arctic multiplet as

$$\begin{aligned} \boldsymbol{\Upsilon}'| &:= \Big(1 + \lambda\zeta\Big)\boldsymbol{\Upsilon}| = \boldsymbol{\Phi}' + \zeta\boldsymbol{\Gamma}' + \sum_{k=2}^{\infty}\boldsymbol{\Upsilon}'_k\zeta^k\,, \\ \boldsymbol{\Phi}' &= \boldsymbol{\Phi}\,, \qquad \boldsymbol{\Gamma}' = \boldsymbol{\Gamma} + \lambda\boldsymbol{\Phi}\,, \qquad \boldsymbol{\Upsilon}'_k = \boldsymbol{\Upsilon}_k| + \lambda\boldsymbol{\Upsilon}_{k-1}| \quad,\ k > 1\ . \end{aligned} \tag{3.24}$$

Here $\boldsymbol{\Gamma}'$ obeys a modified linear constraint of the form:

$$-\frac{1}{4}\left(\bar{\nabla}^2 - 4\mu\right)\boldsymbol{\Gamma}' = \mathrm{i}|\mu|\boldsymbol{\Phi} \ . \tag{3.25}$$

[6]The existence of a duality between the minimal $(\Phi, \bar{\Phi})$ and the non-minimal $(\Gamma, \bar{\Gamma})$ formulations for scalar multiplet became apparent after the foundational work of [5], where these realizations were shown to occur as the compensators corresponding to the old minimal and non-minimal formulations, respectively, for $\mathcal{N}=1$ supergravity.

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Such a constraint is typical of chiral-non-minimal multiplets [33]. The complex superfields $\boldsymbol{\Upsilon}'_k$ with $k > 1$ are obviously unconstrained. Now, the contour integral in (3.21) can easily be performed, and the auxiliary fields integrated out, whence the action $S_{\text{non-conf}}$ becomes

$$S_{\text{non-conf}} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, E\Big(\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} - \bar{\boldsymbol{\Gamma}}'\boldsymbol{\Gamma}'\Big)\ . \tag{3.26}$$

The second SUSY and U(1) transformations of this action are:

$$\begin{aligned}\delta_\varepsilon \boldsymbol{\Phi} &= \mathrm{i}\varepsilon|\mu|\boldsymbol{\Phi} - \big(\bar\varepsilon_{\dot\alpha}\bar\nabla^{\dot\alpha} + \varepsilon\mu\big)\,\boldsymbol{\Gamma}' = -\frac{1}{4}(\bar\nabla^2 - 4\mu)(\varepsilon\boldsymbol{\Gamma}')\,,\\ \delta_\varepsilon \boldsymbol{\Gamma}' &= \mathrm{i}\varepsilon|\mu|\boldsymbol{\Gamma}' + (\varepsilon^\alpha\nabla_\alpha + \varepsilon\bar\mu)\,\boldsymbol{\Phi}\ .\end{aligned} \tag{3.27}$$

The generalized complex linear superfield $\boldsymbol{\Gamma}'$, which is constrained by (3.25), can be dualized into a covariantly chiral scalar $\boldsymbol{\Psi}$, $\bar\nabla^{\dot\alpha}\boldsymbol{\Psi} = 0$, to result with the following purely chiral action:

$$S^{(\text{dual})}_{\text{non-conf}} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, E\bigg(\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} + \bar{\boldsymbol{\Psi}}\boldsymbol{\Psi} - \mathrm{i}\frac{\bar\mu}{|\mu|}\boldsymbol{\Psi}\boldsymbol{\Phi} + \mathrm{i}\frac{\mu}{|\mu|}\bar{\boldsymbol{\Psi}}\bar{\boldsymbol{\Phi}}\bigg)\ . \tag{3.28}$$

In a flat superspace limit, $\mu \to 0$, the last two terms in (3.28) will drop out. The second SUSY and U(1) transformations of the model (3.28) coincide, modulo a simple re-labeling of the chiral variables, with (3.19).

The difference between the hypermultiplet models (3.13) and (3.20) can naturally be understood in terms of their dual tensor multiplet models. The conformal theory (3.13) turns out to be dual to the improved $\mathcal{N} = 2$ tensor model [11, 32, 34, 35]. When realized in the $\mathcal{N} = 2$ AdS superspace, the latter is described by the following Lagrangian:

$$\mathcal{L}^{++}_{\text{impr.-tensor}} = -G^{++}\ln\frac{G^{++}}{\boldsymbol{S}^{++}}\,, \tag{3.29}$$

with G^{++} a real $O(2)$ multiplet. The non-conformal theory (3.20) is dual to the tensor multiplet model

$$\mathcal{L}^{++}_{\text{tensor}} = -\frac{1}{2}\frac{(G^{++})^2}{\boldsymbol{S}^{++}}\ . \tag{3.30}$$

This is similar to the situation in $\mathcal{N} = 1$ AdS supersymmetry, where the conformal scalar multiplet model described by the Lagrangian

$$L_{\text{conf}} = \bar\Phi\Phi \tag{3.31}$$

is dual to the improved tensor multiplet model [36]

$$L_{\text{impr.-tensor}} = -G\ln G\,, \qquad \big(\bar\nabla^2 - 4\mu\big)\,G = 0\,, \qquad G = \bar G\,, \tag{3.32}$$

while the non-conformal scalar multiplet model

$$L_{\text{non-conf}} = \frac{1}{2}\big(\bar{\boldsymbol{\Phi}} + \boldsymbol{\Phi}\big)^2 \tag{3.33}$$

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is dual to the ordinary tensor multiplet model [37]

$$L_{\text{tensor}} = -\frac{1}{2}G^2 \ . \tag{3.34}$$

A nonlinear generalization of the tensor multiplet model (3.30) is

$$\mathcal{L}^{++} = \boldsymbol{S}^{++}\, F\left(\frac{G^{++}}{\boldsymbol{S}^{++}}\right), \tag{3.35}$$

for some function F, compare with the rigid $\mathcal{N}=2$ supersymmetric models for tensor multiplets [11]. This theory can be seen to be dual to a weight-zero polar multiplet model of the form

$$\mathcal{L}^{++} = \boldsymbol{S}^{++}\, \mathbb{F}\Big(\widetilde{\boldsymbol{\Upsilon}} + \boldsymbol{\Upsilon}\Big), \tag{3.36}$$

for some function $\mathbb{F}$ related to F.

3.3 Models involving the intrinsic vector multiplet

The structure of off-shell vector multiplets in a background of $\mathcal{N}=2$ conformal supergravity is discussed in [1]; see also appendix B. In the case of AdS$^{4|8}$, we have $\mathcal{S}^{ij} = \bar{\mathcal{S}}^{ij} = \boldsymbol{S}^{ij}$. Then, the Bianchi identity for the field strength $\mathcal{W}$ of an Abelian vector multiplet, eq. (B.2), tells us that there exists a vector multiplet with a constant field strength, $\mathcal{W}_0$, which can be chosen to be

$$\mathcal{W}_0 = 1 \ . \tag{3.37}$$

Its existence is supported by the geometry of the AdS superspace, and for this reason this vector multiplet will be called intrinsic. We denote the corresponding tropical prepotential by $V_0(z,u^+)$, and it should be emphasized that V_0 is defined modulo gauge transformations of the form:

$$\delta V_0 = \lambda + \widetilde{\lambda}\,, \tag{3.38}$$

where λ is a covariant weight-zero arctic multiplet. Using V_0 allows us to construct a number of interesting models in AdS$^{4|8}$.

Consider a system of Abelian vector supermultiplets in AdS$^{4|8}$ described by their covariantly chiral field strengths $\mathcal{W}_I$, where $I = 1,\dots,n$. The dynamics of this system can be described by a Lagrangian of the form:

$$\mathcal{L}^{++} = -\frac{1}{4}V_0\Big[\Big((\mathcal{D}^+)^2 + 4\boldsymbol{S}^{++}\Big)\mathcal{F}(\mathcal{W}_I) + \Big((\bar{\mathcal{D}}^+)^2 + 4\boldsymbol{S}^{++}\Big)\bar{\mathcal{F}}(\bar{\mathcal{W}}_I)\Big], \tag{3.39}$$

with $\mathcal{F}(\mathcal{W}_I)$ a holomorphic function. The action generated by $\mathcal{L}^{++}$ is invariant under the gauge transformations (3.38). This theory is an AdS extension of the famous vector multiplet model behind the concept of rigid special geometry [38]. The Lagrangian (3.39) is analogous to the rigid harmonic superspace representation for effective vector multiplet models given in [39]. In section 5, we will return to a study of the model (3.39) for the case when AdS$^{4|8}$ is replaced by a general conformally flat superspace.

To describe massive hypermultiplets, we can follow the construction originally developed in the $\mathcal{N}=2$ super-Poincaré case within the harmonic superspace approach [40] and later generalized to the projective superspace [41, 42]. That is, off-shell hypermultiplets should simply be coupled to the intrinsic vector multiplet, following the general pattern of coupling polar hypermultiplets to vector multiplets [12]. A massive weight-one polar hypermultiplet can be described by the Lagrangian

$$\mathcal{L}_1^{++} = \mathrm{i}\,\widetilde{\Upsilon}^+ \mathrm{e}^{m\,V_0} \Upsilon^+ , \qquad m = \mathrm{const} \tag{3.40}$$

which is invariant under the gauge transformation of V_0, eq. (3.38), accompanied by

$$\delta\Upsilon^+ = -m\,\lambda\,\Upsilon^+ \ . \tag{3.41}$$

Similarly, a massive weight-zero polar multiplet can be described by the gauge-invariant Lagrangian:

$$\mathcal{L}_2^{++} = \frac{\boldsymbol{S}^{++}}{|\boldsymbol{S}|}\widetilde{\boldsymbol{\Upsilon}} \mathrm{e}^{m\,V_0} \boldsymbol{\Upsilon} , \qquad m = \mathrm{const} \ . \tag{3.42}$$

4. Conformal flatness and intrinsic vector multiplet

We have seen that the dynamics of various models in $\mathrm{AdS}^{4|8}$ is formulated using the pre-potential of the intrinsic vector multiplet. To reduce such actions to the $\mathcal{N} = 1$ AdS superspace, it is advantageous to realize $\mathrm{AdS}^{4|8}$ as a conformally flat superspace.

The fact that the $\mathcal{N} = 2$ AdS superspace is locally conformal flat has already been discussed in the literature [43]. This result will be re-derived in a more general setting in subsection 4.1.

It is useful to start by recalling the structure of super-Weyl transformations in 4D $\mathcal{N} = 2$ conformal supergravity following [1]. The superspace geometry describing the 4D $\mathcal{N} = 2$ Weyl multiplet was studied in detail in [1], and a summary is given in appendix A. The corresponding covariant derivatives $\mathcal{D}_{\underline{A}} = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_i^{\dot\alpha})$ obey the constraints (A.8), and the latter are solved in terms of the dimension 1 tensors $\mathcal{S}^{ij}$, $\mathcal{G}_{\alpha\dot\alpha}$, $\mathcal{Y}_{\alpha\beta}$ and $\mathcal{W}_{\alpha\beta}$ and their complex conjugates, see eqs. (A.9a)–(A.9c). Let $D_{\underline{A}} = (D_a, D_\alpha^i, \bar{D}_i^{\dot\alpha})$ be another set of covariant derivatives satisfying the same constraints (A.8), with S^{ij}, $G_{\alpha\dot\alpha}$, $Y_{\alpha\beta}$ and $W_{\alpha\beta}$ being the dimension 1 components of the torsion. The two supergeometries, which are associated with $\mathcal{D}_A$ and D_A, are said to be conformally related (equivalently, they describe the same Weyl multiplet) if they are related by a super-Weyl transformation of the form:[7]

$$\mathcal{D}_\alpha^i = \mathrm{e}^{\frac12\bar\sigma}\Big(D_\alpha^i + (D^{\gamma i}\sigma)M_{\gamma\alpha} - (D_{\alpha k}\sigma)J^{ki}\Big) , \tag{4.1a}$$

$$\bar{\mathcal{D}}_{\dot\alpha i} = \mathrm{e}^{\frac12\sigma}\Big(\bar{D}_{\dot\alpha i} + (\bar{D}_i^{\dot\gamma}\bar\sigma)\bar{M}_{\dot\gamma\dot\alpha} + (\bar{D}_{\dot\alpha}^k\bar\sigma)J_{ki}\Big) , \tag{4.1b}$$

$$\begin{aligned}\mathcal{D}_a = \mathrm{e}^{\frac12(\sigma+\bar\sigma)}\Big(& D_a + \frac{\mathrm{i}}{4}(\sigma_a)^\alpha{}_{\dot\beta}(\bar{D}_k^{\dot\beta}\bar\sigma)D_\alpha^k + \frac{\mathrm{i}}{4}(\sigma_a)^\alpha{}_{\dot\beta}(D_\alpha^k\sigma)\bar{D}_k^{\dot\beta} - \frac12\big(D^b(\sigma+\bar\sigma)\big)M_{ab} \\ & + \frac{\mathrm{i}}{8}(\tilde\sigma_a)^{\dot\alpha\alpha}(D^{\beta k}\sigma)(\bar{D}_{\dot\alpha k}\bar\sigma)M_{\alpha\beta} + \frac{\mathrm{i}}{8}(\tilde\sigma_a)^{\dot\alpha\alpha}(\bar{D}_k^{\dot\beta}\bar\sigma)(D_\alpha^k\sigma)\bar{M}_{\dot\alpha\dot\beta} \\ & - \frac{\mathrm{i}}{4}(\tilde\sigma_a)^{\dot\alpha\alpha}(D_\alpha^k\sigma)(\bar{D}_{\dot\alpha}^l\bar\sigma)J_{kl}\Big) ,\end{aligned} \tag{4.1c}$$

[7] In [1], only the infinitesimal super-Weyl transformation was given.

where the parameter σ is covariantly chiral $\bar{D}^{\dot\alpha}_i \sigma = 0$. The dimension-1 components of the torsion are related as follows:

$$\mathcal{S}_{ij} = \mathrm{e}^{\bar\sigma}\Big(S_{ij} - \frac{1}{4}(D^\gamma_{(i} D_{\gamma j)}\sigma) + \frac{1}{4}(D^\gamma_{(i}\sigma)(D_{\gamma j)}\sigma)\Big)\,, \tag{4.2a}$$

$$\mathcal{G}_\alpha{}^{\dot\beta} = \mathrm{e}^{\frac{1}{2}(\sigma+\bar\sigma)}\Big(G_\alpha{}^{\dot\beta} - \frac{\mathrm{i}}{4}(\sigma^c)_\alpha{}^{\dot\beta} D_c(\sigma-\bar\sigma) - \frac{1}{8}(D^k_\alpha\sigma)(\bar{D}^{\dot\beta}_k\bar\sigma)\Big)\,, \tag{4.2b}$$

$$\mathcal{Y}_{\alpha\beta} = \mathrm{e}^{\bar\sigma}\Big(Y_{\alpha\beta} - \frac{1}{4}(D^k_{(\alpha} D_{\beta)k}\sigma) - \frac{1}{4}(D^k_{(\alpha}\sigma)(D_{\beta)k}\sigma)\Big)\,, \tag{4.2c}$$

$$\mathcal{W}_{\alpha\beta} = \mathrm{e}^{\sigma} W_{\alpha\beta}\ . \tag{4.2d}$$

The geometry $\mathcal{D}_{\underline{A}}$ will be called *conformally flat* if the covariant derivatives $D_{\underline{A}}$ correspond to a flat superspace.

Consider a vector multiplet. With respect to the conformally related covariant derivatives $\mathcal{D}_{\underline{A}}$ and $D_{\underline{A}}$, it is characterized by different covariantly chiral field strengths $\mathcal{W}$ and W obeying the equations:

$$\bar{\mathcal{D}}^i_{\dot\alpha}\mathcal{W} = 0\,, \qquad \big(\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_\gamma + 4\mathcal{S}^{ij}\big)\mathcal{W} = \big(\bar{\mathcal{D}}^{(i}_{\dot\gamma}\bar{\mathcal{D}}^{j)\dot\gamma} + 4\bar{\mathcal{S}}^{ij}\big)\bar{\mathcal{W}}\,,$$
$$\bar{D}^i_{\dot\alpha} W = 0\,, \qquad \big(D^{\gamma(i}D^{j)}_\gamma + 4S^{ij}\big)W = \big(\bar{D}^{(i}_{\dot\gamma}\bar{D}^{j)\dot\gamma} + 4\bar{S}^{ij}\big)\bar{W}\ .$$

The field strengths are related to each other as follows [1]:

$$\mathcal{W} = \mathrm{e}^{\sigma}\, W\ . \tag{4.3}$$

Consider a covariant weight-n projective supermultiplet. With respect to the conformally related covariant derivatives $\mathcal{D}_{\underline{A}}$ and $D_{\underline{A}}$, it is described by superfields $\mathcal{Q}^{(n)}$ and $Q^{(n)}$ obeying the constraints

$$\mathcal{D}^+_\alpha \mathcal{Q}^{(n)} = \bar{\mathcal{D}}^+_{\dot\alpha}\mathcal{Q}^{(n)} = 0\,, \qquad D^+_\alpha Q^{(n)} = \bar{D}^+_{\dot\alpha} Q^{(n)} = 0\ . \tag{4.4}$$

In the case of matter multiplets, these superfields are related to each other as follows[8] [1]:

$$\mathcal{Q}^{(n)} = \mathrm{e}^{\frac{n}{2}(\sigma+\bar\sigma)}\, Q^{(n)}\ . \tag{4.5}$$

As argued in [1], the super-Weyl gauge freedom can always be used to impose the reality condition $\mathcal{S}_{ij} = \bar{\mathcal{S}}_{ij}$. The same condition can be chosen for the supergeometry generated by the covariant derivatives $D_{\underline{A}}$. Therefore, if the conformally related supergeometries are characterized by the reality conditions

$$\mathcal{S}_{ij} = \bar{\mathcal{S}}_{ij}\,, \qquad S_{ij} = \bar{S}_{ij}\,, \tag{4.6}$$

then eq. (4.2a) tells us that

$$W := \mathrm{e}^{-\sigma} \tag{4.7}$$

[8]The super-Weyl transformation laws (4.3) and (4.5) have natural counterparts in the case of 5D $\mathcal{N} = 1$ supergravity [3].

is the covariantly chiral field strength of a vector multiplet with respect to the covariant derivatives $D_{\underline{A}}$. Due to (4.3), we then have $\mathcal{W} = 1$.

It is instructive to compare the $\mathcal{N} = 2$ super-Weyl transformation, eqs. (4.1a)–(4.1c), with that in $\mathcal{N} = 1$ old minimal supergavity [44]:

$$\nabla_\alpha = F D_\alpha - 2(D^\gamma F) M_{\gamma\alpha} , \qquad F := \varphi^{1/2}\bar{\varphi}^{-1} , \qquad \bar{D}^{\dot{\alpha}}\varphi = 0 \tag{4.8a}$$

$$\bar{\nabla}_{\dot{\alpha}} = \bar{F}\bar{D}_{\dot{\alpha}} - 2(\bar{D}^{\dot{\gamma}}\bar{F})\bar{M}_{\dot{\gamma}\dot{\alpha}} , \tag{4.8b}$$

$$\nabla_{\alpha\dot{\alpha}} = \frac{\mathrm{i}}{2}\{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\} . \tag{4.8c}$$

Here $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^{\dot{\alpha}})$ and $D_A = (D_a, D_\alpha, \bar{D}^{\dot{\alpha}})$ are two sets of $\mathcal{N} = 1$ supergravity covariant derivatives obeying the modified Wess-Zumino constraints.

4.1 Reconstructing the intrinsic vector multiplet

The superspace geometry of $\mathrm{AdS}^{4|8}$ is determined by the relations (2.1) and (2.2). Let us demonstrate that $\mathrm{AdS}^{4|8}$ is conformally flat, which we note would imply that $S_{ij} = G_{\alpha\dot{\alpha}} = Y_{\alpha\beta} = W_{\alpha\beta} = 0$ in eqs. (4.2a)–(4.2d). Our first task is to search for a chiral scalar σ such that $\mathcal{Y}_{\alpha\beta} = \mathcal{G}_{\alpha\dot{\beta}} = 0$. The equation $\mathcal{Y}_{\alpha\beta} = 0$ is equivalent to

$$D^k_{(\alpha} D_{\beta)k}\, \mathrm{e}^{\sigma} = 0 . \tag{4.9}$$

The equation $\mathcal{G}_{\alpha\dot{\beta}} = 0$ is equivalent to

$$[D^k_\alpha, \bar{D}^{\dot{\alpha}}_k]\mathrm{e}^{\sigma+\bar{\sigma}} = 0 . \tag{4.10}$$

The covariant derivatives of the flat global $\mathcal{N} = 2$ superspace are $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i)$, with

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} - \mathrm{i}(\sigma^b)_\alpha{}^{\dot{\beta}}\bar{\theta}^i_{\dot{\beta}}\partial_b , \qquad \bar{D}^{\dot{\alpha}}_i = \frac{\partial}{\partial\bar{\theta}^i_{\dot{\alpha}}} - \mathrm{i}(\sigma^b)_\beta{}^{\dot{\alpha}}\theta^\beta_i\partial_b . \tag{4.11}$$

Consider a Lorentz invariant ansatz for σ and $\bar{\sigma}$ given by

$$\mathrm{e}^{\sigma} = A(x^2_{\mathrm{L}}) + \theta_{ij}B^{ij}(x^2_{\mathrm{L}}) + \theta^4 C(x^2_{\mathrm{L}}) , \qquad \mathrm{e}^{\bar{\sigma}} = \bar{A}(x^2_{\mathrm{R}}) + \bar{\theta}^{ij}\bar{B}_{ij}(x^2_{\mathrm{R}}) + \bar{\theta}^4\bar{C}(x^2_{\mathrm{R}}) , \tag{4.12}$$

where

$$x^a_{\mathrm{L}} := x^a - \mathrm{i}(\sigma^a)_\alpha{}^{\dot{\alpha}}\theta^\alpha_k\bar{\theta}^k_{\dot{\alpha}} , \qquad \theta_{ij} := \theta^\alpha_i\theta_{\alpha j} , \qquad \theta^4 := \theta_{ij}\theta^{ij} , \tag{4.13a}$$

$$x^a_{\mathrm{R}} := x^a + \mathrm{i}(\sigma^a)_\alpha{}^{\dot{\alpha}}\theta^\alpha_k\bar{\theta}^k_{\dot{\alpha}} , \qquad \bar{\theta}^{ij} := \bar{\theta}^i_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}j} , \qquad \bar{\theta}^4 := \bar{\theta}^{ij}\bar{\theta}_{ij} , \tag{4.13b}$$

and the functions $\bar{A}$, $\bar{B}_{ij}$, $\bar{C}$ are the complex conjugates of A, B^{ij}, C. The variables x^a_{L} and θ^α_i parametrize the chiral subspace of the flat $\mathcal{N} = 2$ superspace.

Equation (4.9) proves to restrict the coefficients in (4.12) to look like

$$A(x^2) = a_1 + a_2\, x^2 , \qquad B^{ij}(x^2) = b^{ij} , \qquad C(x^2) = 0 , \tag{4.14}$$

where a_1, a_2, b^{ij} are constant parameters. Next, equation (4.10) imposes additional conditions on the parameters a_1, a_2 and b^{ij}:

$$a_1\bar{a}_2 = \bar{a}_1 a_2 , \qquad b^{ik}\bar{b}_{kj} = -4a_1\bar{a}_2\delta^i_j . \tag{4.15}$$

Without loss of generality, the constant a_1 can be chosen to be $a_1 = 1$, and then the relations (4.15) are equivalent to

$$b^{ij} = q\boldsymbol{s}^{ij}\,, \quad a_2 = -\frac{1}{4}\boldsymbol{s}^2\,, \qquad \overline{\boldsymbol{s}^{ij}} = \boldsymbol{s}_{ij}\,, \quad |q| = 1\,, \quad \boldsymbol{s}^2 := \frac{1}{2}\boldsymbol{s}^{ij}\boldsymbol{s}_{ij}\ . \tag{4.16}$$

It can be seen that the parameter q coincides with that appearing in (2.2). In accordance with the consideration in section 2, we set $q = 1$. Now, the solution to eqs. (4.9) and (4.10) can be expressed as

$$\mathrm{e}^{\sigma} = 1 - \frac{1}{4}\boldsymbol{s}^2 x_{\mathrm{L}}^2 + \boldsymbol{s}^{ij}\theta_{ij}\,, \qquad \mathrm{e}^{\bar\sigma} = 1 - \frac{1}{4}\boldsymbol{s}^2 x_{\mathrm{R}}^2 + \boldsymbol{s}_{ij}\bar\theta^{ij}\ . \tag{4.17}$$

Note that the tensors $\mathcal{S}^{ij}$ and $\bar{\mathcal{S}}^{ij}$ are expressed in terms of σ and $\bar\sigma$ as follows:

$$\mathcal{S}^{ij} = \frac{1}{4}\mathrm{e}^{\sigma+\bar\sigma}(D^{ij}\mathrm{e}^{-\sigma})\,, \qquad \bar{\mathcal{S}}^{ij} = \frac{1}{4}\mathrm{e}^{\sigma+\bar\sigma}(\bar D^{ij}\mathrm{e}^{-\bar\sigma})\,, \tag{4.18}$$

with $D^{ij} := D^{\gamma(i}D^{j)}_{\gamma}$ and $\bar D^{ij} := \bar D^{(i}_{\dot\gamma}\bar D^{j)\dot\gamma}$. It also holds

$$\mathcal{S}^{ij} = \boldsymbol{s}^{ij} \ + \ O(\theta)\,, \qquad \bar{\mathcal{S}}^{ij} = \boldsymbol{s}^{ij} \ + \ O(\theta)\ . \tag{4.19}$$

Then, the relation

$$\mathcal{S}^{ij} = \bar{\mathcal{S}}^{ij} \equiv \boldsymbol{S}^{ij} \tag{4.20}$$

holds as a consequence of the Bianchi identities. Defining a new chiral superfield

$$W_0 := \mathrm{e}^{-\sigma} = \left(1 - \frac{1}{4}\boldsymbol{s}^2 x_{\mathrm{L}}^2 + \boldsymbol{s}^{ij}\theta_{ij}\right)^{-1}\,, \qquad \bar D^{\dot\alpha}_i W_0 = 0\,, \tag{4.21}$$

one can see that eq. (4.20) is equivalent to

$$D^{ij}W_0 = \bar D^{ij}\bar W_0\ . \tag{4.22}$$

This is the Bianchi identity for the field strength of an Abelian vector multiplet in flat superspace [45]. It is an instructive exercise to check eq. (4.22) by explicit calculations.

It follows from the expression for W_0, eq. (4.21), and the explicit form for the vector covariant derivative $\mathcal{D}_a$, eq. (4.1c), that the space-time metric is

$$\mathrm{d}s^2 = \mathrm{d}x^a\,\mathrm{d}x_a\left(W_0\bar W_0\right)\Big|_{\theta=0} = \frac{\mathrm{d}x^a\mathrm{d}\,x_a}{\left(1 - \frac{1}{4}\boldsymbol{s}^2x^2\right)^2}\ . \tag{4.23}$$

Modulo a trivial redefinition, this expression coincides with the metric in the north chart of AdS_4 defined in appendix D, with x^a being the stereographic coordinates. The metric can be brought to the form (D.3) by re-scaling $x^a \to 2x^a$ and then identifying $\boldsymbol{s}^2 = R^{-2}$. As expected, the conformally flat representation (4.1a)–(4.1c) is defined only locally.

Associated with the field strengths W_0 and $\bar W_0$ is their descendant

$$\Sigma_0^{ij} := \frac{1}{4}D^{ij}W_0 = \frac{1}{4}\bar D^{ij}\bar W_0\,, \qquad \overline{\Sigma_0^{ij}} = \varepsilon_{ik}\varepsilon_{jl}\,\Sigma_0^{kl} \tag{4.24}$$

enjoying the properties

$$D_\alpha^{(i}\Sigma_0^{jk)} = \bar{D}^{\dot\alpha(i}\Sigma_0^{jk)} = 0 \qquad (4.25)$$

that are characteristic of the $\mathcal{N}=2$ tensor multiplet. Contracting the indices of Σ_0^{ij} with the isotwistor variables $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$, we then obtain the following real $O(2)$ multiplet:

$$\Sigma_0^{++}(z,u^+) := u_i^+ u_j^+ \Sigma_0^{ij}(z)\,, \qquad D_\alpha^+\Sigma_0^{++} = \bar{D}^{\dot\alpha+}\Sigma_0^{++} = 0\ . \qquad (4.26)$$

It can be shown that Σ_0^{++} has the form:

$$\Sigma_0^{++} = \frac{\boldsymbol{s}^{++}}{\left(1-\frac{1}{4}\boldsymbol{s}^2x_{\rm A}^2\right)^2} - \frac{2\boldsymbol{s}^2\big((\theta^+)^2+(\bar\theta^+)^2\big)}{\left(1-\frac{1}{4}\boldsymbol{s}^2x_{\rm A}^2\right)^3} - \frac{2{\rm i}\boldsymbol{s}^2\boldsymbol{s}^{+-}(x_{\rm A})_\alpha{}^{\dot\alpha}\theta^{\alpha+}\bar\theta^+_{\dot\alpha}}{(u^+u^-)\left(1-\frac{1}{4}\boldsymbol{s}^2x_{\rm A}^2\right)^3}$$
$$+\frac{1}{2}\frac{\boldsymbol{s}^2\boldsymbol{s}^{--}\big(8+\boldsymbol{s}^2x_{\rm A}^2\big)(\theta^+)^2(\bar\theta^+)^2}{(u^+u^-)^2\left(1-\frac{1}{4}\boldsymbol{s}^2x_{\rm A}^2\right)^4}\ . \qquad (4.27)$$

Here $\boldsymbol{s}^{\pm\pm} = \boldsymbol{s}^{ij}u_i^\pm u_j^\pm$, $\theta_\alpha^\pm = \theta_\alpha^i u_i^\pm$ and $\bar\theta_{\dot\alpha}^\pm = \bar\theta_{\dot\alpha}^i u_i^\pm$, $(\theta^+)^2 = \theta^{+\alpha}\theta_\alpha^+$ and

$$x_{\rm A}^a = x^a + \frac{\rm i}{(u^+u^-)}(\sigma^a)_\alpha{}^{\dot\alpha}\left(\theta^{\alpha+}\bar\theta^-_{\dot\alpha} + \theta^{\alpha-}\bar\theta^+_{\dot\alpha}\right)\ . \qquad (4.28)$$

The variables $x_{\rm A}^a$, θ_α^+ and $\bar\theta_{\dot\alpha}^+$ are annihilated by the covariant derivateves $D_\alpha^+ := u_i^+ D_\alpha^i$ and $\bar{D}^{\dot\alpha+} := u_i^+\bar{D}^{\dot\alpha i}$, and can be used to parametrize the analytic subspace of harmonic superspace [24, 25]. One can check that Σ_0^{++} has the form (4.26), and hence does not depend on u^-,

$$\frac{\partial}{\partial u^-}\Sigma_0^{++} = 0\,, \qquad (4.29)$$

in spite of the fact that separate contributions to the right-hand side of (4.27) explicitly depend on u^-. In conclusion, we give the explicit expression for the torsion $\boldsymbol{S}^{ij}$:

$$\boldsymbol{S}^{ij} = (W_0\bar W_0)^{-1}\,\Sigma_0^{ij}\ . \qquad (4.30)$$

It is important to point out that now $\boldsymbol{S}^{ij}$ is covariantly constant, $\mathcal{D}_\alpha^i\boldsymbol{S}^{kl} = \bar{\mathcal{D}}_{\dot\alpha}^i\boldsymbol{S}^{kl} = 0$, but not constant. This clearly differs from the analysis in section 2, and the origin of this disparity is very simple. In section 2, we imposed the SU(2) gauge (2.4) in which only a U(1) part of the SU(2) connection survived, and the covariant derivatives had the form (2.21). Here we are using the conformally flat representation for the covariant derivatives, eqs. (4.1a) and (4.1b), such that the connection becomes a linear combination of all the generators of the group SU(2).

4.2 Prepotential for the intrinsic vector multiplet

The field strength W_0 of the intrinsic vector multiplet, eqs. (4.17) and (4.21), depends on the *constant* isotensor $\boldsymbol{s}^{ij} = \boldsymbol{s}^{ji}$ obeying the reality condition $\overline{\boldsymbol{s}^{ij}} = \boldsymbol{s}_{ij}$. By applying a rigid SU(2) rotation one can always set

$$\boldsymbol{s}^{\underline{12}} = 0\ . \qquad (4.31)$$

This choice will be used in the remainder of the paper.

Modulo gauge transformations, the prepotential for the intrinsic vector multiplet can be chosen to be

$$V_0(z,u^+) = V_0(z,\zeta) = {\rm i}\frac{\boldsymbol{\theta}^2(\zeta)+\bar{\boldsymbol{\theta}}^2(\zeta)}{\zeta\Big(1-\frac14|\boldsymbol{s}^{\underline{11}}|^2x_{\rm A}^2(\zeta)\Big)} - {\rm i}\frac{\big(\zeta\boldsymbol{s}^{\underline{11}}+\frac{1}{\zeta}\boldsymbol{s}^{\underline{22}}\big)\boldsymbol{\theta}^2(\zeta)\bar{\boldsymbol{\theta}}^2(\zeta)}{\zeta^2\Big(1-\frac14|\boldsymbol{s}^{\underline{11}}|^2x_{\rm A}^2(\zeta)\Big)^2}\ . \tag{4.32}$$

Here we have made use of the complex coordinate ζ for $\mathbb{C}P^1$ as well as the following ζ-dependent superspace variables

$$\begin{aligned}\boldsymbol{\theta}^\alpha(\zeta) &= -\zeta\theta^\alpha_{\underline{2}}-\theta^\alpha_{\underline{1}}\,, \qquad \bar{\boldsymbol{\theta}}_{\dot\alpha}(\zeta) = -\zeta\bar\theta^{\underline{1}}_{\dot\alpha}+\bar\theta^{\underline{2}}_{\dot\alpha}\,,\\ x^a_{\rm A}(\zeta) &= x^a+{\rm i}(\sigma^a)_\alpha{}^{\dot\alpha}\boldsymbol{\theta}^\alpha(\zeta)\bar\theta^{\underline{1}}_{\dot\alpha}+{\rm i}(\sigma^a)_\alpha{}^{\dot\alpha}\theta^\alpha_{\underline{2}}\bar{\boldsymbol{\theta}}_{\dot\alpha}(\zeta)\,,\end{aligned} \tag{4.33}$$

which are annihilated by $\zeta_i D^i_\alpha$ and $\zeta_i\bar D^{\dot\alpha i}$, with $\zeta_i=(-\zeta,1)$.

4.3 $\mathcal{N}=1$ reduction revisited

We have elaborated upon the superspace reduction $\mathcal{N}=2\to\mathcal{N}=1$ in subsection 2.2 using the representation (2.21) for the covariant derivatives. Such a reduction should be carried out afresh if the covariant derivatives are given in the conformally flat representation defined by eqs. (4.1a) and (4.1b). One of the reasons for this is that the component $\boldsymbol{S}^{\underline{12}}$ of the torsion $\boldsymbol{S}^{ij}$ does not vanish and the algebra of the operators $(\mathcal{D}_a,\mathcal{D}^{\underline{1}}_\alpha,\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}})$ is no longer closed, for the third relation in (2.23) turns into

$$[\mathcal{D}_a,\mathcal{D}^{\underline{1}}_\beta] = \frac{\rm i}{2}(\sigma_a)_{\beta\dot\gamma}\boldsymbol{S}^{\underline{11}}\bar{\mathcal{D}}^{\dot\gamma}_{\underline{1}}+\frac{\rm i}{2}(\sigma_a)_{\beta\dot\gamma}\boldsymbol{S}^{\underline{12}}\bar{\mathcal{D}}^{\dot\gamma}_{\underline{2}}\ . \tag{4.34}$$

Nevertheless, it can be shown, using (4.31), that the projection of $\boldsymbol{S}^{\underline{12}}$ does vanish,

$$\boldsymbol{S}^{\underline{12}}|=0\ . \tag{4.35}$$

Another consequence of the choice (4.31) is

$$(D^{\underline{2}}_\alpha\sigma)| = (\bar D^{\dot\alpha}_{\underline{2}}\bar\sigma)| = 0\ . \tag{4.36}$$

Then, applying the $\mathcal{N}=1$ projection to the covariant derivatives,

$$\mathcal{D}_{\underline{A}}| := \mathcal{E}_{\underline{A}}{}^{\underline{M}}|\partial_{\underline{M}}]+\frac12\Omega_{\underline{A}}{}^{bc}|M_{bc}+\Phi_{\underline{A}}{}^{kl}|J_{kl}\,, \tag{4.37}$$

for $\mathcal{D}^{\underline{1}}_\alpha|$ and $\bar{\mathcal{D}}_{\dot\alpha\underline{1}}|$ we get

$$\mathcal{D}^{\underline{1}}_\alpha| = {\rm e}^{\frac12\bar\sigma|}\Big(D_\alpha+(D^\gamma\sigma|)M_{\gamma\alpha}+(D_\alpha\sigma|)J_{\underline{12}}\Big)\,, \tag{4.38a}$$

$$\bar{\mathcal{D}}_{\dot\alpha\underline{1}}| = {\rm e}^{\frac12\sigma|}\Big(\bar D_{\dot\alpha}+(\bar D^{\dot\gamma}\bar\sigma|)\bar M_{\dot\gamma\dot\alpha}+(\bar D_{\dot\alpha}\bar\sigma|)J_{\underline{12}}\Big)\ . \tag{4.38b}$$

Here D_α and $\bar D^{\dot\alpha}$ are the spinor covariant derivatives for the flat global $\mathcal{N}=1$ superspace parametrized by $(x^a,\theta^\alpha,\bar\theta_{\dot\alpha})$, with

$$\theta^\alpha := \theta^\alpha_{\underline{1}}\,,\quad \bar\theta_{\dot\alpha} := \bar\theta^{\underline{1}}_{\dot\alpha}\,,\qquad D_\alpha := D^{\underline{1}}_\alpha|\,,\quad \bar D^{\dot\alpha} := \bar D^{\dot\alpha}_{\underline{1}}|\ . \tag{4.39}$$

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As is seen from (4.38a) and (4.38b), the operators $\mathcal{D}^{\underline{1}}_{\alpha}|$ and $\bar{\mathcal{D}}_{\dot\alpha\underline{1}}|$ do not involve any partial derivatives with respect to $\theta_{\underline{2}}$ and $\bar\theta^{\underline{2}}$. Another important property is that the operator $J_{\underline{12}}$ is diagonal when acting on $\mathcal{D}^{\underline{1}}_{\alpha}$ and $\bar{\mathcal{D}}_{\dot\alpha\underline{1}}$. Therefore, for any positive integer k, it holds that $\big(\mathcal{D}_{\hat\alpha_1}\cdots\mathcal{D}_{\hat\alpha_k}U\big)\big| = \mathcal{D}_{\hat\alpha_1}|\cdots\mathcal{D}_{\hat\alpha_k}|U|$, where $\mathcal{D}_{\hat\alpha} = (\mathcal{D}^{\underline{1}}_{\alpha}, \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}})$ and U is an arbitrary superfield. This implies that the operators $(\mathcal{D}_a|, \mathcal{D}^{\underline{1}}_{\alpha}|, \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}|)$ satisfy the (anti-)commutation relations:

$$
\begin{aligned}
\big\{\mathcal{D}^{\underline{1}}_{\alpha}|, \mathcal{D}^{\underline{1}}_{\beta}|\big\} &= 4\boldsymbol{S}^{\underline{11}}|M_{\alpha\beta}\,, \qquad \big\{\mathcal{D}^{\underline{1}}_{\alpha}|, \bar{\mathcal{D}}^{\dot\beta}_{\underline{1}}|\big\} = -2\mathrm{i}(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c|\,, \\
\big[\mathcal{D}_a|, \mathcal{D}^{\underline{1}}_{\beta}|\big] &= \frac{\mathrm{i}}{2}(\sigma_a)_{\beta\dot\gamma}\boldsymbol{S}^{\underline{11}}|\bar{\mathcal{D}}^{\dot\gamma}_{\underline{1}}|\,.
\end{aligned}
\tag{4.40}
$$

The algebra (4.40) is isomorphic to that of the $\mathcal{N}=1$ AdS covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$, see appendix C. Unlike ∇_A, however, the operators $(\mathcal{D}_a|, \mathcal{D}^{\underline{1}}_{\alpha}|, \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}|)$ involve a zero-curvature U(1) connection, with $J_{\underline{12}}$ the U(1) generator. The latter connection can be gauged away. Making use of the explicit action of the generator $J_{\underline{12}}$ on the covariant derivatives,

$$
[J_{\underline{12}}, \mathcal{D}^{\underline{1}}_{\alpha}] = -\frac{1}{2}\mathcal{D}^{\underline{1}}_{\alpha}\,, \qquad [J_{\underline{12}}, \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}] = \frac{1}{2}\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}\,.
\tag{4.41}
$$

one finds

$$
\mathrm{e}^{-(\bar\sigma-\sigma)|J_{\underline{12}}}\,\mathcal{D}^{\underline{1}}_{\alpha}|\,\mathrm{e}^{(\bar\sigma-\sigma)|J_{\underline{12}}} = \nabla_\alpha\,, \qquad \mathrm{e}^{-(\bar\sigma-\sigma)|J_{\underline{12}}}\,\bar{\mathcal{D}}_{\dot\alpha\underline{1}}|\,\mathrm{e}^{(\bar\sigma-\sigma)|J_{\underline{12}}} = \bar\nabla_{\dot\alpha}\,.
\tag{4.42}
$$

Here the operators $\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$ have the form (4.8a)–(4.8c), where $D_A = (D_a, D_\alpha, \bar D^{\dot\alpha})$ are the flat $\mathcal{N}=1$ covariant derivatives, and the chiral superfield φ is

$$
\varphi := W_0| = \left(1 - \frac{\mu\bar\mu}{4}x_{\mathrm{L}}^2 - \bar\mu\theta^2\right)^{-1}, \qquad \bar D^{\dot\alpha}\varphi = 0\,,
\tag{4.43}
$$

with

$$
\bar\mu := -\boldsymbol{s}^{\underline{11}}\,, \qquad \mu := -\boldsymbol{s}^{\underline{22}} = -\boldsymbol{s}_{\underline{11}}\,.
\tag{4.44}
$$

The operators $\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$ coincide with the $\mathcal{N}=1$ AdS covariant derivatives as given in [7], and satisfy the (anti-)commutation relations (C.2a) and (C.2b).

Let us describe the action of the U(1)-rotation $\mathrm{e}^{-(\bar\sigma-\sigma)J_{\underline{12}}}$ on different types of projective multiplets. For a covariant weight-n arctic hypermultiplet (2.39) it holds

$$
\Upsilon^{[n]}(z,\zeta) = \sum_{k=0}^{+\infty}\Upsilon_k(z)\zeta^k\,, \qquad \mathrm{e}^{-(\bar\sigma-\sigma)J_{\underline{12}}}\Upsilon^{[n]}(z,\zeta) = \mathrm{e}^{-\frac{n}{2}(\bar\sigma-\sigma)}\Upsilon^{[n]}\big(z, \mathrm{e}^{(\bar\sigma-\sigma)}\zeta\big)\,.
\tag{4.45}
$$

Here we have used the results of [1] for the SU(2)-transformation rules of the component superfields of projective multiplets. In the case of a real weight-2n projective superfield (2.43), such as $O(2n)$ multiplets, one finds

$$
R^{[2n]}(z,\zeta) = \sum_{k=-\infty}^{+\infty}R_k(z)\zeta^k\,, \qquad \mathrm{e}^{-(\bar\sigma-\sigma)J_{\underline{12}}}R^{[2n]}(z,\zeta) = R^{[2n]}\big(z, \mathrm{e}^{(\bar\sigma-\sigma)}\zeta\big)\,.
\tag{4.46}
$$

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To conclude this section, we wish to give the expressions for $\boldsymbol{S}^{ij}|$ and $V_0|$ which will be useful in what follows. For the $O(2)$ multiplet $\boldsymbol{S}^{++} := u^+_i u^+_j \boldsymbol{S}^{ij}$, one can show

$$\boldsymbol{S}^{++}| = \mathrm{i} u^{+\underline{1}} u^{+\underline{2}} \boldsymbol{S}(\zeta)| , \qquad \boldsymbol{S}(\zeta)| = \mathrm{i}\Big(\varphi^{-1}\bar{\varphi}\,\bar{\mu}\,\zeta + \bar{\varphi}^{-1}\varphi\,\mu\,\frac{1}{\zeta}\Big) . \tag{4.47}$$

It is important to note that

$$\mathrm{e}^{-(\bar{\sigma}-\sigma)|J_{\underline{12}}} \boldsymbol{S}(\zeta)| = \mathrm{i}\left(\bar{\mu}\,\zeta + \mu\,\frac{1}{\zeta}\right) , \tag{4.48}$$

where we have used (4.46). For the prepotential $V_0(\zeta)$ of the intrinsic vector multiplet, we obtain

$$V_0(\zeta)| = \mathrm{i}\left(\varphi\bar{\theta}^2\,\zeta + \bar{\varphi}\theta^2\,\frac{1}{\zeta}\right) , \tag{4.49}$$

and hence

$$\hat{V}_0(\zeta) := \mathrm{e}^{-(\bar{\sigma}-\sigma)J_{\underline{12}}} V_0(\zeta)| = \mathrm{i}\left(\big(\varphi^2\bar{\varphi}^{-1}\bar{\theta}^2\big)\zeta + \big(\bar{\varphi}^2\varphi^{-1}\theta^2\big)\frac{1}{\zeta}\right) := \zeta V_+ - \frac{1}{\zeta}V_- \,. \tag{4.50}$$

4.4 $\mathcal{N}=2$ AdS Killing supervectors: II

In this subsection, the $\mathcal{N}=2$ AdS Killing supervectors are explicitly evaluated using the conformally flat representation for $\mathcal{D}_{\underline{A}}$ derived earlier.

Our starting point will be the observation that the conformally related supergeometries have isomorphic superconformal algebras (see [7] for a pedagogical discussion of this result in the case of 4D $\mathcal{N}=1$ supergravity). Therefore, since the superspaces $\mathbb{R}^{4|8}$ and $\mathrm{AdS}^{4|8}$ are conformally related, they possess the same superconformal algebra, su$(2,2|2)$. It is well known how su$(2,2|2)$ is realized in the 4D $\mathcal{N}=2$ flat superspace, see e.g. [28, 29, 46 – 48] and references therein. Let us first recall this realization following [28, 29, 48].

By definition, a superconformal Killing vector of $\mathbb{R}^{4|8}$

$$\boldsymbol{\xi} = \bar{\boldsymbol{\xi}} = \boldsymbol{\xi}^{\underline{A}}(z) D_{\underline{A}} = \boldsymbol{\xi}^a \partial_a + \boldsymbol{\xi}^\alpha_i D^i_\alpha + \bar{\boldsymbol{\xi}}^i_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}_i \tag{4.51}$$

obeys the constraint

$$\delta_{\boldsymbol{\sigma}} D_{\underline{A}} + \Big[\boldsymbol{\xi} + \frac{1}{2}K^{cd}M_{cd} + K^{kl}J_{kl}, D_{\underline{A}}\Big] = 0\,, \tag{4.52}$$

for a chiral scalar $\boldsymbol{\sigma}(z)$, $\bar{D}^{\dot{\alpha}}_i \boldsymbol{\sigma} = 0$, which generates an infinitesimal super-Weyl transformation, a real antisymmetric tensor $K^{cd}(z)$ and a real symmetric tensor $K^{kl}(z)$. This constraint implies

$$\bar{D}^{\dot{\alpha}}_i K^{\beta\gamma} = 0\,, \qquad D^i_\alpha K^{\beta\gamma} = \delta^{(\beta}_\alpha D^{\gamma)i}\boldsymbol{\sigma}\,, \qquad D^i_\alpha K^{kl} = \varepsilon^{i(k} D^{l)}_\alpha \boldsymbol{\sigma}\,, \tag{4.53}$$

as well as

$$[\boldsymbol{\xi}, D^i_\alpha] = -\frac{1}{2}\bar{\boldsymbol{\sigma}}\, D^i_\alpha - K_\alpha{}^\beta D^i_\beta - K^i{}_j D^j_\alpha \,. \tag{4.54}$$

The latter equation, in turn, leads to

$$K_{\alpha\beta} = \frac{1}{2}\, D^i_{(\alpha}\boldsymbol{\xi}_{\beta)i}\,, \qquad \boldsymbol{\sigma} = \frac{1}{2}\bar{D}^{\dot{\alpha}}_i\bar{\boldsymbol{\xi}}^i_{\dot{\alpha}}\,, \tag{4.55a}$$

$$K^i{}_j = \frac{1}{2}\Big(D^i_\alpha\boldsymbol{\xi}^\alpha_j - \frac{1}{2}\delta^i_j D^k_\alpha\boldsymbol{\xi}^\alpha_k\Big) = -\frac{1}{2}\Big(\bar{D}^{\dot{\alpha}}_j\bar{\boldsymbol{\xi}}^i_{\dot{\alpha}} - \frac{1}{2}\delta^i_j\bar{D}^{\dot{\alpha}}_k\bar{\boldsymbol{\xi}}^k_{\dot{\alpha}}\Big)\,, \tag{4.55b}$$

as well as

$$\bar{D}^{\dot{\alpha}}_i\boldsymbol{\xi}^{\dot{\beta}\beta} = 4\mathrm{i}\varepsilon^{\dot{\alpha}\dot{\beta}}\boldsymbol{\xi}^\beta_i\,, \qquad \bar{D}^{\dot{\alpha}}_i\boldsymbol{\xi}^\beta_j = 0\,. \tag{4.56}$$

The general expression for the superconformal Killing vector can be shown to be

$$\begin{aligned}
\boldsymbol{\xi}^a &= \frac{1}{2}\left(\boldsymbol{\xi}^a_{\mathrm{L}} + \bar{\boldsymbol{\xi}}^a_{\mathrm{R}}\right) + \mathrm{i}(\sigma^a)_\alpha{}^{\dot{\alpha}}\boldsymbol{\xi}^\alpha_k\bar{\theta}^k_{\dot{\alpha}} + \mathrm{i}(\sigma^a)_\alpha{}^{\dot{\alpha}}\bar{\boldsymbol{\xi}}^k_{\dot{\alpha}}\theta^\alpha_k\,,\\
\boldsymbol{\xi}^{\dot{\alpha}\alpha}_{\mathrm{L}} &= p^{\dot{\alpha}\alpha} + (r+\bar{r})x^{\dot{\alpha}\alpha}_{\mathrm{L}} - \bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}}x^{\dot{\beta}\alpha}_{\mathrm{L}} - x^{\dot{\alpha}\beta}_{\mathrm{L}}\omega_\beta{}^\alpha + x^{\dot{\alpha}\beta}_{\mathrm{L}}k_{\beta\dot{\beta}}x^{\dot{\beta}\alpha}_{\mathrm{L}} + 4\mathrm{i}\bar{\epsilon}^{\dot{\alpha}k}\theta^\alpha_k - 4x^{\dot{\alpha}\beta}_{\mathrm{L}}\eta^k_\beta\theta^\alpha_k\,,\\
\boldsymbol{\xi}^\alpha_i &= \epsilon^\alpha_i + \bar{r}\theta^\alpha_i - \theta^\beta_i\omega_\beta{}^\alpha - \Lambda_i{}^j\theta^\alpha_j + \theta^\beta_i k_{\beta\dot{\beta}}x^{\dot{\beta}\alpha}_{\mathrm{L}} - \mathrm{i}\bar{\eta}_{i\dot{\beta}}x^{\dot{\beta}\alpha}_{\mathrm{L}} - 4\theta^\beta_i\eta^k_\beta\theta^\alpha_k\,,
\end{aligned} \tag{4.57}$$

see, e.g., [47, 28] for two different derivations. Here the constant parameters $(\omega_\alpha{}^\beta,\ \bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}})$ correspond to a Lorentz transformation, $p^{\dot{\alpha}\beta}$ a space-time translation, $k_{\alpha\dot{\beta}}$ a special conformal transformation, r a combined scale and chiral $\mathrm{U}(1)$ transformation, $(\epsilon^\alpha_i,\ \bar{\epsilon}^{\dot{\alpha}i})$ and $(\eta^i_\alpha,\ \bar{\eta}_{i\dot{\alpha}})$ Q-supersymmetry and S-supersymmetry transformations respectively, and finally $\Lambda_i{}^j$ an $\mathrm{SU}(2)$ transformation.

If W is the chiral field strength of an Abelian vector multiplet in $\mathbb{R}^{4|8}$, such that $D^{\alpha i}D^j_\alpha W = \bar{D}^i_{\dot{\alpha}}\bar{D}^{j\dot{\alpha}}\bar{W}$ is the corresponding Bianchi identity, its superconformal transformation is

$$\delta W = \boldsymbol{\xi} W + \boldsymbol{\sigma} W\,, \tag{4.58}$$

see, e.g., [48]. The superconformal transformations of the rigid projective multiplets are given in [29].

Now, let us return to the $\mathcal{N}=2$ AdS superspace, and let $\xi^{\underline{A}}(z)\mathcal{E}_{\underline{A}}$ be its Killing supervector. We can represent

$$\xi^{\underline{A}}(z)\mathcal{E}_{\underline{A}} = \boldsymbol{\xi}^{\underline{A}}(z)D_{\underline{A}} \equiv \boldsymbol{\xi}\,, \tag{4.59}$$

where

$$\boldsymbol{\xi}^a = \mathrm{e}^{\frac{1}{2}(\sigma+\bar{\sigma})}\xi^a\,, \qquad \boldsymbol{\xi}^\alpha_i = \mathrm{e}^{\frac{1}{2}\bar{\sigma}}\xi^\alpha_i + \frac{\mathrm{i}}{4}\mathrm{e}^{\frac{1}{2}(\sigma+\bar{\sigma})}\xi^\alpha{}_{\dot{\beta}}(\bar{D}^{\dot{\beta}}_i\bar{\sigma})\,. \tag{4.60}$$

Then, eq. (2.8) proves to be equivalent to the fact that $\boldsymbol{\xi}$ is a superconformal Killing supervector in $\mathbb{R}^{4|8}$ such that

$$\delta W_0 = \boldsymbol{\xi} W_0 + \boldsymbol{\sigma} W_0 = 0\,, \tag{4.61}$$

with W_0 the field strength of the intrinsic vector multiplet. In other words, W_0 is invariant under the $\mathcal{N}=2$ AdS transformations (which is completely natural, keeping in mind that $\mathcal{W}_0 = 1$). The invariance of W_0 implies that the AdS transformation of the prepotential V_0 is a pure gauge transformation.

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The general solution of (4.61) can be shown to be

$$r = 0\,, \tag{4.62a}$$

$$k^a = \frac{1}{4}\boldsymbol{s}^2 p^a\,, \tag{4.62b}$$

$$\eta^i_\alpha = \frac{1}{2}\boldsymbol{s}^{ij}\epsilon_{\alpha j}\,, \qquad \bar\eta_i^{\dot\alpha} = \frac{1}{2}\boldsymbol{s}_{ij}\bar\epsilon^{\dot\alpha j}\,, \tag{4.62c}$$

$$\Lambda_{ij} = l\boldsymbol{s}_{ij}\,, \qquad \bar l = l\,, \tag{4.62d}$$

with no restrictions on the Lorentz parameters. Using the solution (4.62a)–(4.62d) in (4.57), from (4.60) one can read off the $\mathcal{N}=2$ AdS Killing supervectors ξ in terms of $\boldsymbol{\xi}$.

It is instructive to consider the $\mathcal{N}=1$ reduction of the $\mathcal{N}=2$ AdS Killing supervectors. Let us first give the $\mathcal{N}=1$ projection of the superconformal Killing vector $\boldsymbol{\xi}$ associated with the $\mathcal{N}=2$ AdS Killing vector field $\xi^{\underline{A}}(z)\mathcal{E}_{\underline{A}}$:

$$\begin{aligned}\boldsymbol{\lambda}^{\alpha\dot\alpha} &= \boldsymbol{\xi}^{\alpha\dot\alpha}| = \Big(1-\frac{|\mu|^2}{4}\theta^2\bar\theta^2\Big)p^{\alpha\dot\alpha} + \frac{|\mu|^2}{4}x^{\alpha\dot\beta}p_{\beta\dot\beta}x^{\beta\dot\alpha} - \omega^\alpha{}_\beta x^{\beta\dot\alpha} - \bar\omega^{\dot\alpha}{}_{\dot\beta}x^{\alpha\dot\beta}\\ &\quad - 2\theta^\alpha(2\mathrm{i}\bar\epsilon^{\dot\alpha\underline{1}} + \bar\mu x^{\beta\dot\alpha}\epsilon_{\beta\underline{1}}) - 2\bar\theta^{\dot\alpha}(2\mathrm{i}\epsilon^\alpha_{\underline{1}} - \mu x^{\alpha\dot\beta}\bar\epsilon^{\underline{1}}_{\dot\beta}) - \mathrm{i}\theta^\alpha\bar\theta_{\dot\beta}\Big(2\bar\omega^{\dot\alpha\dot\beta} + \frac{|\mu|^2}{2}p^{\beta(\dot\alpha}x_\beta{}^{\dot\beta)}\Big)\\ &\quad - \mathrm{i}\bar\theta^{\dot\alpha}\theta_\beta\Big(2\omega^{\alpha\beta} - \frac{|\mu|^2}{2}p^{(\alpha}{}_{\dot\beta}x^{\beta)\dot\beta}\Big) - 2\mathrm{i}\bar\mu\epsilon^\alpha_{\underline{1}}\bar\theta^{\dot\alpha}\theta^2 - 2\mathrm{i}\mu\bar\epsilon^{\dot\alpha\underline{1}}\theta^\alpha\bar\theta^2\,,\end{aligned} \tag{4.63a}$$

$$\boldsymbol{\lambda}^\alpha = \boldsymbol{\xi}^\alpha_{\underline{1}}| = \epsilon^\alpha_{\underline{1}}\big(1-\bar\mu\theta^2\big) - \theta^\beta\omega_\beta{}^\alpha + \frac{|\mu|^2}{4}\theta^\beta p_{\beta\dot\beta}x_L^{\dot\beta\alpha} + \frac{\mathrm{i}}{2}\mu\bar\epsilon^{\underline{1}}_{\dot\beta}x_L^{\dot\beta\alpha}\,, \tag{4.63b}$$

$$\boldsymbol{\varepsilon}^\alpha = \boldsymbol{\xi}^\alpha_{\underline{2}}| = \epsilon^\alpha_{\underline{2}} + l\bar\mu\theta^\alpha + \frac{\mathrm{i}}{2}\bar\mu\bar\epsilon^{\underline{2}}_{\dot\beta}x_L^{\dot\beta\alpha}\ . \tag{4.63c}$$

Then, the $\mathcal{N}=1$ AdS Killing supervector $\Lambda = \lambda^a\nabla_a + \lambda^\alpha\nabla_\alpha + \bar\lambda_{\dot\alpha}\bar\nabla^{\dot\alpha}$ is expressed in terms of $\boldsymbol{\lambda}_a$ and $\boldsymbol{\lambda}^\alpha$ as follows:

$$\lambda^a = \varphi^{\frac{1}{2}}\bar\varphi^{\frac{1}{2}}\boldsymbol{\lambda}^a\,, \tag{4.64a}$$

$$\lambda^\alpha = -\frac{\mathrm{i}}{8}\bar\nabla_{\dot\beta}\lambda^{\alpha\dot\beta} = \varphi^{-\frac{1}{2}}\bar\varphi\Big(\boldsymbol{\lambda}^\alpha + \frac{\mathrm{i}}{4}\boldsymbol{\lambda}^\alpha{}_{\dot\beta}\bar D^{\dot\beta}\log\bar\varphi\Big)\ . \tag{4.64b}$$

These expressions agree with [7]. The second supersymmetry and U(1) transformations in the $\mathcal{N}=1$ AdS superspace are generated by ε and ε^α which are related to $\boldsymbol{\varepsilon}^\alpha$ appearing in eq. (4.63c) as follows:

$$\varepsilon^\alpha = \varphi^{\frac{1}{2}}\boldsymbol{\varepsilon}^\alpha\,, \tag{4.65a}$$

$$\begin{aligned}\varepsilon &= \frac{1}{2\bar\mu}\nabla^\alpha\varepsilon_\alpha = \frac{1}{2\bar\mu}\varphi\bar\varphi^{-1}\Big(D^\alpha\boldsymbol{\varepsilon}_\alpha + 2(D^\alpha\log\varphi)\boldsymbol{\varepsilon}_\alpha\Big)\,,\\ &= \frac{1}{2\mu}\bar\nabla_{\dot\alpha}\bar\varepsilon^{\dot\alpha} = \frac{1}{2\mu}\bar\varphi\varphi^{-1}\Big(\bar D_{\dot\alpha}\bar{\boldsymbol{\varepsilon}}^{\dot\alpha} + 2(\bar D_{\dot\alpha}\log\bar\varphi)\bar{\boldsymbol{\varepsilon}}^{\dot\alpha}\Big)\ .\end{aligned} \tag{4.65b}$$

The explicit expression for ε is

$$\begin{aligned}\varepsilon &= -l + \frac{(2-\mu\bar\theta^2)\epsilon_{\underline{2}}\theta + (2-\bar\mu\theta^2)\bar\epsilon^{\underline{2}}\bar\theta + \mathrm{i}x^a(\mu\epsilon_{\underline{2}}\sigma_a\bar\theta - \bar\mu\theta\sigma_a\bar\epsilon^{\underline{2}}) + l(\bar\mu\theta^2+\mu\bar\theta^2)}{\Big(1-\frac{|\mu|^2}{4}x^2\Big)}\\ &\quad + \frac{\mu\epsilon_{\underline{2}}\theta\bar\theta^2 + \bar\mu\theta^2\bar\epsilon^{\underline{2}}\bar\theta + \frac{\mathrm{i}|\mu|^2}{2}x^a(\epsilon_{\underline{2}}\sigma_a\bar\theta\theta^2 - \theta\sigma_a\bar\epsilon^{\underline{2}}\bar\theta^2) + l|\mu|^2\theta^2\bar\theta^2}{\Big(1-\frac{|\mu|^2}{4}x^2\Big)^2}\ .\end{aligned} \tag{4.66}$$

As argued earlier, the $\mathcal{N}=2$ AdS transformation of the prepotential V_0 is a pure gauge transformation. Any AdS transformation should be accompanied by the inverse of the associated gauge transformation, in order to keep V_0 fixed. This will result in modified supersymmetry transformations of charged hypermultiplets (supersymmetry with central charge), in complete analogy with the rigid supersymmetric case [40]. Here we provide the expression for the induced gauge transformation of $\hat{V}_0| = \mathrm{e}^{-(\bar{\sigma}-\sigma)J_{\underline{12}}}V_0|$, see eq. (4.50). A direct calculation gives

$$\delta\hat{V}_0| = \lambda| + \tilde{\lambda}|\,, \qquad \lambda| = \lambda_0| + \zeta\lambda_1| + \zeta^2\lambda_2|\,, \tag{4.67}$$

where

$$\lambda_0| = \mathrm{i}\frac{2\epsilon_{\underline{2}}\theta - \mathrm{i}\bar{\mu}x_L^a\theta\sigma_a\bar{\epsilon}^{\underline{2}} + l\bar{\mu}\theta^2}{\left(1-\frac{|\mu|^2}{4}x_L^2\right)}\,, \quad \lambda_1| = \Lambda V_+\,, \quad \lambda_2| = \left(\varepsilon^\alpha\nabla_\alpha - \varepsilon\bar{\mu}\right)V_+\,. \tag{4.68}$$

Note that in eq. (4.67), $\lambda_0|$ is chiral and $\lambda_1|$ can be seen to be complex linear, $(\bar{\nabla}^2-4\mu)\lambda_1 = 0$. This agrees with the requirement that the gauge parameter λ should be a weight-zero arctic superfield.

5. Dynamics in $\mathcal{N}=2$ conformally flat superspace

In this section we study supersymmetric theories in an arbitrary conformally flat $\mathcal{N}=2$ superspace $\mathcal{M}^{4|8}$. The corresponding covariant derivatives $\mathcal{D}_{\underline{A}}$ will be assumed to have the form (4.1a)–(4.1c), with $D_{\underline{A}}$ the covariant derivatives for $\mathbb{R}^{4|8}$. It will also be assumed that the torsion tensor $\mathcal{S}_{ij}$ is real, $\mathcal{S}_{ij} = \bar{\mathcal{S}}_{ij}$. The latter property means that $W_0 := \mathrm{e}^{-\sigma}$ is the field strength of an Abelian vector multiplet, that is the intrinsic vector multiplet for $\mathcal{M}^{4|8}$.

For our subsequent consideration, it will be useful to view conformally flat $\mathcal{N}=2$ supergeometry as a conformally flat $\mathcal{N}=1$ superspace endowed with an Abelian $\mathcal{N}=1$ vector multiplet. Indeed, for the covariant derivatives (4.1a)–(4.1c), it holds that

$$\mathrm{e}^{-(\bar{\sigma}-\sigma)|J_{\underline{12}}}\,\mathcal{D}^{\underline{1}}_\alpha|\,\mathrm{e}^{(\bar{\sigma}-\sigma)|J_{\underline{12}}} = \nabla_\alpha + 2\mathrm{i}\,\mathcal{W}_{0\alpha}J_{\underline{22}}\,, \tag{5.1a}$$

$$\mathrm{e}^{-(\bar{\sigma}-\sigma)|J_{\underline{12}}}\,\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}}|\,\mathrm{e}^{(\bar{\sigma}-\sigma)|J_{\underline{12}}} = \bar{\nabla}^{\dot{\alpha}} - 2\mathrm{i}\,\bar{\mathcal{W}}_0^{\dot{\alpha}}J_{\underline{11}}\,. \tag{5.1b}$$

Here the operators $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^{\dot{\alpha}})$ have the form (4.8a)–(4.8c), where $D_A = (D_a, D_\alpha, \bar{D}^{\dot{\alpha}})$ are the flat $\mathcal{N}=1$ covariant derivatives, and the chiral superfield φ is defined as

$$\varphi := W_0|\,, \qquad \bar{D}^{\dot{\alpha}}\varphi = 0\,. \tag{5.2}$$

The spinor superfield in (5.1a), $\mathcal{W}_0^\alpha$, is the covariantly chiral field strength of an Abelian $\mathcal{N}=1$ vector multiplet,

$$\bar{\nabla}_{\dot{\alpha}}\mathcal{W}_0^\alpha = 0\,, \qquad \nabla^\alpha\mathcal{W}_{0\alpha} = \bar{\nabla}_{\dot{\alpha}}\bar{\mathcal{W}}_0^{\dot{\alpha}}\,. \tag{5.3}$$

and is related to W_0 as follows:

$$\mathcal{W}_{0\alpha} = \varphi^{-3/2}W_{0\alpha}\,, \qquad W_{0\alpha} := -\frac{\mathrm{i}}{2}D^{\underline{2}}_\alpha W_0|\,. \tag{5.4}$$

In the case of $\mathcal{N}=2$ AdS superspace, φ is given by eq. (4.43) and $\mathcal{W}_{0\alpha}=0$.

In accordance with [1], off-shell hypermultiplets are described by covariant arctic superfields of weight n, $\Upsilon^{(n)}(u^+)$, and their smile-conjugates. Given such a superfield in $\mathcal{M}^{4|8}$, we can use the standard representation $\Upsilon^{(n)}(u^+)=(u^{+\underline{1}})^n\Upsilon^{[n]}(\zeta)$, and then

$$\mathrm{e}^{-(\bar\sigma-\sigma)J_{\underline{12}}}\Upsilon^{[n]}(\zeta)\Big| = \mathrm{e}^{-\frac{n}{2}(\bar\sigma-\sigma)}\Upsilon^{[n]}\big(\mathrm{e}^{(\bar\sigma-\sigma)}\zeta\big)\Big| \equiv \Phi+\zeta\Gamma+\sum_{k=2}^{+\infty}\zeta^k\hat\Upsilon_k|\ . \tag{5.5}$$

Here the leading components Φ and Γ are covariantly chiral and complex linear, respectively,

$$\bar\nabla^{\dot\alpha}\Phi=0\,,\qquad \big(\bar\nabla^2-4R\big)\,\Gamma=0\,, \tag{5.6}$$

where $R=-(1/4)\varphi^{-2}\bar D^2\bar\varphi$ is the chiral scalar component of the torsion in the $\mathcal{N}=1$ conformally flat superspace, see. e.g. [7] for a review.

5.1 Projecting the $\mathcal{N}=2$ action into $\mathcal{N}=1$ superspace: II

Our first goal is to project the supersymmetric action (1.1) corresponding to $\mathcal{M}^{4|8}$,

$$S=\frac{1}{2\pi}\oint_C(u^+\mathrm{d}u^+)\int\mathrm{d}^4x\,\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\mathcal{E}\,\frac{\mathcal{L}^{++}}{(\mathcal{S}^{++})^2}\,, \tag{5.7}$$

into $\mathcal{N}=1$ superspace. Using the super-Weyl transformation laws given in section 4, for the superfields appearing in (5.7) we find

$$\begin{aligned}\mathcal{L}^{++}&=\mathrm{e}^{\sigma+\bar\sigma}L^{++}\,, &\qquad D^+_\alpha L^{++}=\bar D^+_{\dot\alpha}L^{++}=0\,,\\ \mathcal{S}^{++}&=\mathrm{e}^{\sigma+\bar\sigma}\Sigma_0^{++}\,, &\qquad \mathcal{E}=1\,,\end{aligned} \tag{5.8}$$

where

$$\Sigma_0^{++}=\frac14(D^+)^2W_0=\frac14(\bar D^+)^2\bar W_0\ . \tag{5.9}$$

The new Lagrangian, L^{++}, is a real weight-two projective multiplet in the flat $\mathcal{N}=2$ superspace.

In the action obtained,

$$S=\frac{1}{2\pi}\oint_C(u^+\mathrm{d}u^+)\int\mathrm{d}^4x\,\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\frac{\mathrm{e}^{-\sigma-\bar\sigma}L^{++}}{(\Sigma_0^{++})^2}\,, \tag{5.10}$$

we can make use of the identity

$$(D^+)^4\mathrm{e}^{-\sigma-\bar\sigma}=\Big(\frac14(D^+)^2W_0\Big)\Big(\frac14(\bar D^+)^2\bar W_0\Big)=(\Sigma_0^{++})^2\,,\quad (D^+)^4:=\frac{1}{16}(D^+)^2(\bar D^+)^2\,, \tag{5.11}$$

and then transform (5.10) in the following way:

$$\begin{aligned}S&=\frac{1}{2\pi}\oint_C\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int\mathrm{d}^4x\,(D^-)^4(D^+)^4\,\frac{\mathrm{e}^{-\sigma-\bar\sigma}L^{++}}{(\Sigma_0^{++})^2}\Big|_{\theta=0}\\ &=\frac{1}{2\pi}\oint_C\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int\mathrm{d}^4x\,(D^-)^4\,L^{++}\Big|_{\theta=0}\,,\end{aligned} \tag{5.12}$$

where

$$D^-_\alpha := u^-_i D^i_\alpha\,, \quad \bar D^-_{\dot\alpha} := u^-_i \bar D^i_{\dot\alpha}\,, \qquad (D^-)^4 := \frac{1}{16}(D^-)^2(\bar D^-)^2\ . \tag{5.13}$$

This action can be seen to be invariant under arbitrary projective transformations of the form (B.7). Without loss of generality, we can assume the north pole of $\mathbb{C}P^1$ to be outside of the integration contour, hence u^{+i} can be represented as $u^{+i} = u^{+\underline{1}}(1,\zeta)$, with ζ the local complex coordinate for $\mathbb{C}P^1$. Using the projective invariance (B.7), we can then choose u^-_i to be $u^-_i = (1,0)$. Finally, representing L^{++} in the form

$$L^{++}(z,u^+) = \mathrm{i}\, u^{+\underline{1}}u^{+\underline{2}}\, L(z,\zeta) = \mathrm{i}\big(u^{+\underline{1}}\big)^2\zeta\, L(z,\zeta)\,, \tag{5.14}$$

and also using the fact that L^{++} enjoys the constraints $\zeta_i D^i_\alpha L = \zeta_i \bar D^i_{\dot\alpha} L = 0$, we can finally rewrite S as an integral over the $\mathcal{N}=1$ superspace parametrized by the coordinates: $(x^a, \theta^\alpha_{\underline{1}}, \bar\theta^{\underline{1}}_{\dot\alpha})$. The result is

$$S = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, L(\zeta)\Big|\ . \tag{5.15}$$

As a last step, we replace here $L(\zeta)|$ with the $\mathcal{N}=1$ projection of $\mathcal{L}(\zeta)$ defined as $\mathcal{L}^{++}(u^+) = \mathrm{i}\big(u^{+\underline{1}}\big)^2\zeta\,\mathcal{L}(\zeta)$. Thus

$$\mathcal{L}(\zeta)| = \big(\mathrm{e}^{\sigma+\bar\sigma}L(\zeta)\big)\big| = \frac{1}{\varphi\bar\varphi}L(\zeta)\big|\,, \tag{5.16}$$

and then the action obtained can be rewritten as

$$S = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, \varphi\bar\varphi\,\mathcal{L}(\zeta)\big| = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, E\,\mathcal{L}(\zeta)\big|\ . \tag{5.17}$$

This is the desired $\mathcal{N}=1$ projection of the action (5.7). In the AdS case, the above action coincides with (3.8).

As follows from eqs. (5.1a) and (5.1b), the projection into $\mathcal{N}=1$ superspace should be accompanied by the U(1)-rotation $\mathrm{e}^{-(\bar\sigma-\sigma)|J_{\underline{12}}}$ applied to all superfields. This means that the final expression for the action (5.17) is

$$S = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, E\,\mathcal{L}\Big(\frac{\varphi}{\bar\varphi}\,\zeta\Big)\Big|\ . \tag{5.18}$$

In the rest of this section, the U(1)-rotation $\mathrm{e}^{-(\bar\sigma-\sigma)|J_{\underline{12}}}$ will be assumed to be performed.

5.2 Massive hypermultiplets in AdS$^{4|8}$

As a simple application of the formalism developed, we consider the massive hypermultiplet model (3.42) in AdS$^{4|8}$ (the massive model (3.40) can be studied similarly). The corresponding Lagrangian to be used in (5.18) is

$$\mathcal{L}| = \frac{1}{|\mu|}\boldsymbol{S}(\zeta)\widetilde{\boldsymbol{\Upsilon}}(\zeta)|\mathrm{e}^{mV_0(\zeta)|}\boldsymbol{\Upsilon}(\zeta)|\,, \tag{5.19}$$

We remind that all the superfields are assumed to have been subjected to the U(1)-rotation $\mathrm{e}^{-(\bar\sigma-\sigma)|J_{\underline{12}}}$.

The weight-zero arctic superfield $\boldsymbol{\Upsilon}$ is characterized by the decomposition (3.22). For the prepotential V_0 of the intrinsic vector multiplet, we have

$$\mathrm{e}^{mV_0(\zeta)|} = \Big(1 + m\zeta V_+\Big)\Big(1 - \frac{m}{\zeta}V_-\Big)\,, \quad V_+ = \mathrm{i}\varphi^2\bar{\varphi}^{-1}\,\bar{\theta}^2\,, \quad V_- = -\mathrm{i}\bar{\varphi}^2\varphi^{-1}\,\theta^2\;. \tag{5.20}$$

It is then natural to generalize the superfield redefinition (3.24) to the massive case as follows:

$$\boldsymbol{\Upsilon}'(\zeta)| := (1+\lambda\zeta)\,(1+\zeta V_+)\,\boldsymbol{\Upsilon}(\zeta)|\,, \qquad \boldsymbol{\Upsilon}'(\zeta)| = \boldsymbol{\Phi} + \zeta\boldsymbol{\Gamma}' + \sum_{k=2}^{\infty}\boldsymbol{\Upsilon}'_k\zeta^k\;. \tag{5.21}$$

The component superfield $\boldsymbol{\Gamma}'$ is now constrained by

$$-\frac{1}{4}\,(\bar{\nabla}^2 - 4\mu)\,\boldsymbol{\Gamma}' = \mathrm{i}|\mu|\left(1 + \frac{m}{|\mu|}\right)\boldsymbol{\Phi}\,, \tag{5.22}$$

while the components $\boldsymbol{\Upsilon}'_k$, $k > 1$, are complex unconstrained. Now, the contour integral in the action generated by the Lagrangian (5.19) can easily be performed, and the auxiliary fields integrated out. As a result, the action becomes

$$S = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\Big(\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} - \bar{\boldsymbol{\Gamma}}'\boldsymbol{\Gamma}'\Big)\;. \tag{5.23}$$

It is manifestly $\mathcal{N}=1$ supersymmetric. It also possesses hidden second supersymmetry and U(1) symmetry. These are generated by a real parameter ε under the constraints (2.31), and have the following form:

$$\begin{aligned}
\delta_\varepsilon\boldsymbol{\Phi} &= \mathrm{i}\varepsilon|\mu|\left(1 + \frac{m}{|\mu|}\right)\boldsymbol{\Phi} - \big(\bar{\varepsilon}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}} + \varepsilon\mu\big)\,\boldsymbol{\Gamma}' = -\frac{1}{4}(\bar{\nabla}^2 - 4\mu)(\varepsilon\boldsymbol{\Gamma}')\,,\\
\delta_\varepsilon\boldsymbol{\Gamma}' &= \mathrm{i}\varepsilon|\mu|\left(1 + \frac{m}{|\mu|}\right)\boldsymbol{\Gamma}' + \big(\varepsilon^\alpha\nabla_\alpha + \varepsilon\bar{\mu}\big)\,\boldsymbol{\Phi}\;.
\end{aligned} \tag{5.24}$$

This transformation reduces to (3.27) for $m=0$. A purely chiral action, which is dual to (5.23), proves to be

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\Big(\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} + \bar{\boldsymbol{\Psi}}\boldsymbol{\Psi} - \mathrm{i}\frac{\bar{\mu}}{|\mu|}\Big(1 + \frac{m}{|\mu|}\Big)\boldsymbol{\Psi}\boldsymbol{\Phi} + \mathrm{i}\frac{\mu}{|\mu|}\Big(1 + \frac{m}{|\mu|}\Big)\bar{\boldsymbol{\Psi}}\bar{\boldsymbol{\Phi}}\Big)\;. \tag{5.25}$$

This action reduces to (3.26) for $m=0$. Another interesting special case is $m = -|\mu|$ for which (5.25) turns into the superconformal massless action (3.18).

The symmetry group of (5.25) is OSp(2|4). The second SUSY and U(1) transformations are:

$$\delta_\varepsilon\boldsymbol{\Phi} = -\frac{1}{4}(\bar{\nabla}^2 - 4\mu)(\varepsilon\bar{\boldsymbol{\Psi}})\,, \qquad \delta_\varepsilon\boldsymbol{\Psi} = \frac{1}{4}(\bar{\nabla}^2 - 4\mu)(\varepsilon\bar{\boldsymbol{\Phi}})\;. \tag{5.26}$$

Such transformations are m-independent and identical to those which occur in the different models (3.18). This indicates that the transformations (5.26), in conjunction with the $\mathcal{N}=1$ AdS transformations, form a closed algebra with a central charge proportional to

m. This is indeed the case. One can check that transformations (5.26) have a manifestly $\mathcal{N} = 2$ supersymmetric realization. The latter is given in terms of an isospinor superfield q^i obeying the constraints

$$\mathcal{D}_\alpha^{(i} q^{j)} = \bar{\mathcal{D}}_{\dot\alpha}^{(i} q^{j)} = 0 \tag{5.27}$$

which generalize Sohnius' construction [49] for the off-shell hypermultiplet with intrinsic central charge [50]. Unlike the arctic hypermultiplets (or more general harmonic q^+-hypermultiplets [24, 25]), the above realization can only be used for the construction of simplest supersymmetric theories.

5.3 Vector multiplet self-couplings

We now turn our attention to the system of Abelian vector multiplets described by the Lagrangian

$$\mathcal{L}^{++} = -\frac{1}{4} V_0 \Big[\Big((\mathcal{D}^+)^2 + 4\mathcal{S}^{++} \Big) \mathcal{F}(\mathcal{W}_I) + \Big((\bar{\mathcal{D}}^+)^2 + 4\mathcal{S}^{++} \Big) \bar{\mathcal{F}}(\bar{\mathcal{W}}_I) \Big] , \tag{5.28}$$

In the AdS case, this Lagrangian becomes (3.39). Here we will consider the more general case of an arbitrary conformally flat superspace. We are interested in reducing the model (5.28) to $\mathcal{N} = 1$ conformally flat superspace. Using conformal flatness, it turns out that the dynamics of (5.28) is equivalently described by the Lagrangian

$$L^{++} = -\frac{1}{4} V_0 \left[(D^+)^2 W_0 \mathcal{F}\left(\frac{W_I}{W_0}\right) + (\bar{D}^+)^2 \bar{W}_0 \mathcal{F}\left(\frac{\bar{W}_I}{\bar{W}_0}\right) \right] , \tag{5.29}$$

where

$$\mathcal{W}_I = W_0^{-1} W_I , \qquad \bar{D}^i_{\dot\alpha} W_I = 0 , \qquad D^{ij} W_I = \bar{D}^{ij} \bar{W}_I . \tag{5.30}$$

For the general conformally flat supergeometry, the superfield $W_0 = \mathrm{e}^{-\sigma}$ is only constrained to obey the equation for the field strength of an Abelian vector multiplet in $\mathcal{N} = 2$ flat superspace, and otherwise it is arbitrary. The field strength W_0 is generated by a weight-zero tropical prepotential $V_0(\zeta)$,

$$V_0(\zeta) = \sum_{k=-\infty}^{+\infty} \zeta^k v_k , \qquad \overline{v_k} = (-1)^k v_{-k} , \qquad D^{\underline{1}}_\alpha v_k = D^{\underline{2}}_\alpha v_{k+1} , \tag{5.31}$$

The field strength is given as

$$W_0 = \frac{\mathrm{i}}{4} \bar{D}^2_{\underline{1}} v_1 = \frac{\mathrm{i}}{4} \bar{D}^2_{\underline{2}} v_{-1} . \tag{5.32}$$

The resulting flat-superspace action is

$$S = \frac{\mathrm{i}}{4} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i} \zeta} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \mathrm{d}^2\bar\theta \, V_0(\zeta) \frac{\zeta_i \zeta_j}{\zeta} \left[D^{ij} W_0 \mathcal{F}\left(\frac{W_I}{W_0}\right) + \bar{D}^{ij} \bar{W}_0 \mathcal{F}\left(\frac{\bar{W}_I}{\bar{W}_0}\right) \right] \Big| . \tag{5.33}$$

It involves only the component superfieds v_{-1}, v_0 and v_1 of $V_0(\zeta)$. Computing the contour integral, performing some D-algebra manipulations and using the identities (5.30) and (5.32), one can obtain the equivalent form for the action:

$$
\begin{aligned}
S = &\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\varphi\bar{\varphi}\,\bar{\Phi}_I\mathcal{F}^I(\Phi) + \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\varphi^3\,R\big(2\mathcal{F}(\Phi) - \Phi_I\mathcal{F}^I(\Phi)\big) \\
&+ \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\Big[W_0^\alpha W_{0\alpha}\big(2\mathcal{F}(\Phi) - 2\Phi_I\mathcal{F}^I(\Phi) + \Phi_I\Phi_J\mathcal{F}^{IJ}(\Phi)\big) \\
&\qquad + 2W_0^\alpha W_{I\alpha}\big(\mathcal{F}^I(\Phi) - \Phi_J\mathcal{F}^{IJ}(\Phi)\big) + W_I^\alpha W_{J\alpha}\mathcal{F}^{IJ}(\Phi)\Big] + \text{c.c.}
\end{aligned}
\tag{5.34}
$$

Here we have introduced the $\mathcal{N}=1$ components, Φ_I and $W_{I\alpha}$, of W_I defined as follows:

$$
\varphi\,\Phi_I = W_I\big|\,, \quad W_{I\alpha} := -\frac{\mathrm{i}}{2}D_{\hat{\alpha}}^2 W_I\big|\,, \qquad D^\alpha W_{I\alpha} = \bar{D}_{\dot{\alpha}}\bar{W}_I^{\dot{\alpha}}\,, \tag{5.35}
$$

The similar components of W_0^α are defined in eqs. (5.2) and (5.4). Associated with $W_{I\alpha}$ is the curved-superspace field strength $\mathcal{W}_{\alpha I} = \varphi^{-3/2}W_{\alpha I}$, which obeys the Bianchi identity $\bar{\nabla}^{\dot{\alpha}}\mathcal{W}_{\alpha I} = 0$, $\nabla^\alpha\mathcal{W}_{\alpha I} = \bar{\nabla}_{\dot{\alpha}}\bar{\mathcal{W}}_I^{\dot{\alpha}}$. In terms of the superfields introduced, the action takes the following final form:

$$
\begin{aligned}
S = &\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\bar{\Phi}_I\mathcal{F}^I(\Phi) \\
&+ \int \mathrm{d}^4x\mathrm{d}^4\theta\,\frac{E}{R}\Big[R\big(2\mathcal{F}(\Phi) - \Phi_I\mathcal{F}^I(\Phi)\big) + \mathcal{W}_0^\alpha\mathcal{W}_{0\alpha}\big(2\mathcal{F}(\Phi) - 2\Phi_I\mathcal{F}^I(\Phi) + \Phi_I\Phi_J\mathcal{F}^{IJ}(\Phi)\big) \\
&\qquad + 2\mathcal{W}_0^\alpha\mathcal{W}_{I\alpha}\big(\mathcal{F}^I(\Phi) - \Phi_J\mathcal{F}^{IJ}(\Phi)\big) + \mathcal{W}_I^\alpha\mathcal{W}_{J\alpha}\mathcal{F}^{IJ}(\Phi)\Big] \;+\; \text{c.c.}
\end{aligned}
\tag{5.36}
$$

If $\mathcal{F}(\Phi)$ is a homogeneous function of degree two, $\Phi_I\mathcal{F}^I(\Phi) = 2\mathcal{F}(\Phi)$, the action considerably simplifies, in particular all dependence on $\mathcal{W}_0^\alpha$ disappears,

$$
S = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\bar{\Phi}_I\mathcal{F}^I(\Phi) + \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\frac{E}{R}\mathcal{W}_I^\alpha\mathcal{W}_{J\alpha}\mathcal{F}^{IJ}(\Phi) \;+\; \text{c.c.} \tag{5.37}
$$

The action also simplifies drastically in the case of $\mathrm{AdS}^{4|8}$ where $\mathcal{W}_0^\alpha = 0$.

6. Open problems

To conclude this paper, we would like to list a few interesting open problems.

In the $\mathcal{N}=1$ AdS supersymmetry, there exists a very nice classification of the off-shell superfield types due to Ivanov and Sorin [18] (see also [51] for a review), which is based on their local superprojectors. It would be interesting to carry out a similar analysis for the case of $\mathcal{N}=2$ AdS superspace. This might be useful for deriving a manifestly $\mathcal{N}=2$ supersymmetric formulation for the off-shell higher spin $\mathcal{N}=2$ supermultiplets [21] on AdS^4.

When realizing $\mathrm{AdS}^{4|8}$ as a conformally flat superspace, we used the stereographic coordinates for AdS^4 (defined in appendix D), in which the metric is manifestly $\mathrm{SO}(3,1)$ invariant. By analogy with the five-dimensional consideration of [52], it would be

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interesting to re-do the whole analysis in Poincaré parametrization[9] in which the metric for AdS^4 looks like

$$\mathrm{d}^2 s = \left(\frac{R}{z}\right)^2 \left(\eta_{\hat{m}\hat{n}}\, \mathrm{d}x^{\hat{m}} \mathrm{d}x^{\hat{n}} + \mathrm{d}z^2\right), \qquad R = \mathrm{const}\,, \qquad \hat{m}, \hat{n} = 0, 1, 2, \tag{6.1}$$

with $\eta_{\hat{m}\hat{n}}$ the three-dimensional Minkowski metric. First of all, this would give direct access to three-dimensional superconformal theories. Second, the Poincaré coordinates should be very useful for the explicit elimination of the auxiliary superfields in nonlinear sigma-models of the form (3.4), see [52] for more detail.

It would be desirable to develop harmonic-superspace techniques for $\mathrm{AdS}^{4|8}$. This should proceed similarly to the harmonic-superspace construction developed in the case of 5D $\mathcal{N} = 1$ AdS superspace [26]. The harmonic superspace approach is known to be most suitable for quantum calculations in $\mathcal{N} = 2$ super Yang-Mills theories. Thus it would be very interesting, e.g., to see how the covariant harmonic supergraphs [53, 54] generalize to the AdS case.

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A. Superspace geometry of conformal supergravity

Consider a curved 4D $\mathcal{N} = 2$ superspace $\mathcal{M}^{4|8}$ parametrized by local bosonic (x) and fermionic $(\theta, \bar{\theta})$ coordinates $z^{\underline{M}} = (x^m, \theta^\mu_i, \bar{\theta}^i_{\dot{\mu}})$, where $m = 0, 1, \ldots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $i = \underline{1}, \underline{2}$. The Grassmann variables θ^μ_i and $\bar{\theta}^i_{\dot{\mu}}$ are related to each other by complex conjugation: $\overline{\theta^\mu_i} = \bar{\theta}^{\dot{\mu} i}$. The structure group is chosen to be $\mathrm{SO}(3,1) \times \mathrm{SU}(2)$ [55, 1], and the covariant derivatives $\mathcal{D}_{\underline{A}} = (\mathcal{D}_a, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}_i)$ have the form

$$\mathcal{D}_{\underline{A}} = \mathcal{E}_{\underline{A}} + \Omega_{\underline{A}} + \Phi_{\underline{A}}\,. \tag{A.1}$$

Here $\mathcal{E}_{\underline{A}} = \mathcal{E}_{\underline{A}}{}^{\underline{M}}(z) \partial_{\underline{M}}$ is the supervielbein, with $\partial_{\underline{M}} = \partial / \partial z^{\underline{M}}$,

$$\Omega_{\underline{A}} = \frac{1}{2} \Omega_{\underline{A}}{}^{bc} M_{bc} = \Omega_{\underline{A}}{}^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_{\underline{A}}{}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} \tag{A.2}$$

is the Lorentz connection,

$$\Phi_{\underline{A}} = \Phi_{\underline{A}}{}^{kl} J_{kl}\,, \quad J_{kl} = J_{lk} \tag{A.3}$$

is the SU(2)-connection. The Lorentz generators with vector indices ($M_{ab} = -M_{ba}$) and spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$) are related to each other by the rule:

$$M_{ab} = (\sigma_{ab})^{\alpha\beta} M_{\alpha\beta} - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}\,, \quad M_{\alpha\beta} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} M_{ab}\,, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} M_{ab}\,.$$

[9] Similar to the stereographic coordinates, these coordinates cover one-half of the AdS hyperboloid.

The generators of $\mathrm{SO}(3,1)\times \mathrm{SU}(2)$ act on the covariant derivatives as follows:[10]

$$
\begin{aligned}
&[J_{kl},\mathcal{D}^i_\alpha] = -\delta^i_{(k}\mathcal{D}_{\alpha l)}\,, \qquad [J_{kl},\bar{\mathcal{D}}^{\dot\alpha}_i] = -\varepsilon_{i(k}\bar{\mathcal{D}}^{\dot\alpha}_{l)}\,,\\
&[M_{\alpha\beta},\mathcal{D}^i_\gamma] = \varepsilon_{\gamma(\alpha}\mathcal{D}^i_{\beta)}\,, \qquad [\bar M_{\dot\alpha\dot\beta},\bar{\mathcal{D}}^i_{\dot\gamma}] = \varepsilon_{\dot\gamma(\dot\alpha}\bar{\mathcal{D}}^i_{\dot\beta)}\,, \qquad [M_{ab},\mathcal{D}_c] = 2\eta_{c[a}\mathcal{D}_{b]}\,,
\end{aligned} \tag{A.4}
$$

while $[M_{\alpha\beta},\bar{\mathcal{D}}^i_{\dot\gamma}] = [\bar M_{\dot\alpha\dot\beta},\mathcal{D}^i_\gamma] = [J_{kl},\mathcal{D}_a] = 0$. Our notation and conventions correspond to [7, 1]; they almost coincide with those used in [10] except for the normalization of the Lorentz generators, including a sign in the definition of the sigma-matrices σ_{ab} and $\tilde\sigma_{ab}$.

The supergravity gauge group is generated by local transformations of the form

$$
\delta_K \mathcal{D}_{\underline{A}} = [K,\mathcal{D}_{\underline{A}}]\,, \qquad K = K^{\underline{C}}(z)\mathcal{D}_{\underline{C}} + \frac{1}{2}K^{cd}(z)M_{cd} + K^{kl}(z)J_{kl}\,, \tag{A.5}
$$

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$
\delta_K U = K\,U\ . \tag{A.6}
$$

The covariant derivatives obey (anti-)commutation relations of the form:

$$
[\mathcal{D}_{\underline{A}},\mathcal{D}_{\underline{B}}\} = \mathcal{T}_{\underline{AB}}{}^{\underline{C}}\mathcal{D}_{\underline{C}} + \frac{1}{2}\mathcal{R}_{\underline{AB}}{}^{cd}M_{cd} + \mathcal{R}_{\underline{AB}}{}^{kl}J_{kl}\,, \tag{A.7}
$$

where $\mathcal{T}_{\underline{AB}}{}^{\underline{C}}$ is the torsion, and $\mathcal{R}_{\underline{AB}}{}^{kl}$ and $\mathcal{R}_{\underline{AB}}{}^{cd}$ constitute the curvature. The torsion is subject to the following constraints [55]:

$$
\begin{aligned}
\mathcal{T}^{ij}_{\alpha\beta}{}^{c} &= \mathcal{T}^{ij}_{\alpha\beta k}{}^{\gamma} = \mathcal{T}^{ij}_{\alpha\dot\beta}{}^{k}_{\dot\gamma} = \mathcal{T}^{i\dot\beta}_{\alpha j}{}^{\gamma}_{k} = \mathcal{T}_{a}{}^{j}_{\beta}{}^{c} = \mathcal{T}_{ab}{}^{c} = 0\,,\\
\mathcal{T}^{i\dot\beta}_{\alpha j}{}^{c} &= -2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\,, \qquad \mathcal{T}_{a}{}^{j}_{\beta k}{}^{\gamma} = \delta^j_k\,\mathcal{T}_{a\beta}{}^{\gamma}\ .
\end{aligned} \tag{A.8}
$$

Here we have omitted some constraints which follow by complex conjugation. The algebra of covariant derivatives is [1]

$$
\begin{aligned}
\{\mathcal{D}^i_\alpha,\mathcal{D}^j_\beta\} &= 4\mathcal{S}^{ij}M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\mathcal{Y}^{\gamma\delta}M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{\mathcal{W}}^{\dot\gamma\dot\delta}\bar M_{\dot\gamma\dot\delta}\\
&\quad +2\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathcal{S}^{kl}J_{kl} + 4\mathcal{Y}_{\alpha\beta}J^{ij}\,, && \text{(A.9a)}\\
\{\mathcal{D}^i_\alpha,\bar{\mathcal{D}}^{\dot\beta}_j\} &= -2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c + 4\delta^i_j\mathcal{G}^{\delta\dot\beta}M_{\alpha\delta} + 4\delta^i_j\mathcal{G}_{\alpha\dot\gamma}\bar M^{\dot\gamma\dot\beta} + 8\mathcal{G}_\alpha{}^{\dot\beta}J^i{}_j\,, && \text{(A.9b)}\\
[\mathcal{D}_a,\mathcal{D}^j_\beta] &= \mathrm{i}(\sigma_a)_{(\beta}{}^{\dot\beta}\mathcal{G}_{\gamma)\dot\beta}\mathcal{D}^{\gamma j} + \frac{\mathrm{i}}{2}\Big((\sigma_a)_{\beta\dot\gamma}\mathcal{S}^{jk} - \varepsilon^{jk}(\sigma_a)_\beta{}^{\dot\delta}\bar{\mathcal{W}}_{\dot\delta\dot\gamma} - \varepsilon^{jk}(\sigma_a)^\alpha{}_{\dot\gamma}\mathcal{Y}_{\alpha\beta}\Big)\bar{\mathcal{D}}^{\dot\gamma}_k\\
&\quad + \text{curvature terms}\ . && \text{(A.9c)}
\end{aligned}
$$

Here the real four-vector $\mathcal{G}_{\alpha\dot\alpha}$, the complex symmetric tensors $\mathcal{S}^{ij} = \mathcal{S}^{ji}$, $\mathcal{W}_{\alpha\beta} = \mathcal{W}_{\beta\alpha}$, $\mathcal{Y}_{\alpha\beta} = \mathcal{Y}_{\beta\alpha}$ and their complex conjugates $\bar{\mathcal{S}}_{ij} := \overline{\mathcal{S}^{ij}}$, $\bar{\mathcal{W}}_{\dot\alpha\dot\beta} := \overline{\mathcal{W}_{\alpha\beta}}$, $\bar{\mathcal{Y}}_{\dot\alpha\dot\beta} := \overline{\mathcal{Y}_{\alpha\beta}}$ obey additional differential constraints implied by the Bianchi identities [55, 1]. Of special importance are the following dimension 3/2 identities:

$$
\mathcal{D}^{(i}_\alpha\mathcal{S}^{jk)} = \bar{\mathcal{D}}^{(i}_{\dot\alpha}\mathcal{S}^{jk)} = 0\ . \tag{A.10}
$$

[10] In what follows, the (anti)symmetrization of n indices is defined to include a factor of $(n!)^{-1}$.

B. Vector multiplets in conformal supergravity

Here we discuss the projective-superspace description of off-shell vector multiplets in 4D $\mathcal{N}=2$ conformal supergravity. Following the conventions adopted in [1], an Abelian vector multiplet is described by its field strength $\mathcal{W}(z)$ which is covariantly chiral

$$\bar{\mathcal{D}}^{\dot{\alpha}}_i \mathcal{W} = 0\,, \tag{B.1}$$

and obeys the Bianchi identity

$$\Sigma^{ij} := \frac{1}{4}\Big(\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_{\gamma} + 4\mathcal{S}^{ij}\Big)\mathcal{W} = \frac{1}{4}\Big(\bar{\mathcal{D}}^{(i}_{\dot{\gamma}}\bar{\mathcal{D}}^{j)\dot{\gamma}} + 4\bar{\mathcal{S}}^{ij}\Big)\bar{\mathcal{W}} =: \bar{\Sigma}^{ij}\ . \tag{B.2}$$

Under the infinitesimal super-Weyl transformations, $\mathcal{W}$ varies as

$$\delta_\sigma \mathcal{W} = \sigma \mathcal{W}\ . \tag{B.3}$$

The super-Weyl transformation of Σ^{ij} is

$$\delta_\sigma \Sigma^{ij} = \big(\sigma + \bar{\sigma}\big)\Sigma^{ij}\ . \tag{B.4}$$

The vector multiplet can also be described by its gauge field $\mathcal{V}(z,u^+)$ which is a covariant real weight-zero tropical supermultiplet possessing the following expansion in the north chart of $\mathbb{C}P^1$:

$$\mathcal{D}^+_\alpha \mathcal{V} = \bar{\mathcal{D}}^+_{\dot{\alpha}}\mathcal{V} = 0\,, \qquad \mathcal{V}(z,u^+) = \mathcal{V}(z,\zeta) = \sum_{k=0}^{+\infty}\zeta^k\,\mathcal{V}_k(z)\,, \quad \mathcal{V}_k = (-1)^k\bar{\mathcal{V}}_{-k}\ . \tag{B.5}$$

It turns out that the field strength $\mathcal{W}$ and its conjugate $\bar{\mathcal{W}}$ are expressed in terms of the prepotential V as follows:

$$\mathcal{W}(z) = -\frac{1}{8\pi}\oint\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^2}(\bar{\mathcal{D}}^-_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}-} + 4\bar{\mathcal{S}}^{--})\mathcal{V}(z,u^+)\,, \tag{B.6a}$$

$$\bar{\mathcal{W}}(z) = -\frac{1}{8\pi}\oint\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^2}(\mathcal{D}^{\alpha-}\mathcal{D}^-_{\alpha} + 4\mathcal{S}^{--})\mathcal{V}(z,u^+)\,, \tag{B.6b}$$

with the contour integral being carried out around the origin. These expressions can be shown to be invariant under arbitrary projective transformations of the form:

$$(u_i{}^-\,,\,u_i{}^+) \ \to\ (u_i{}^-\,,\,u_i{}^+)\,R\,, \qquad R = \begin{pmatrix} a & 0\\ b & c\end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})\ . \tag{B.7}$$

Using the fact that $\mathcal{V}(z,u^+)$ is a covariant projective supermultiplet, $\mathcal{D}^+_\alpha\mathcal{V} = \bar{\mathcal{D}}^+_{\dot{\alpha}}\mathcal{V} = 0$, one can show that the right-hand side of (B.6a) is covariantly chiral. For this, it is advantageous to make use of the following equivalent representations:

$$\begin{aligned}
\mathcal{W} &= \frac{1}{8\pi}\oint\frac{\mathrm{d}\zeta}{\zeta^2}\,(\bar{\mathcal{D}}_{\dot{\alpha}\underline{1}}\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} + 4\bar{\mathcal{S}}_{\underline{11}})\,\mathcal{V}(\zeta) = \frac{\mathrm{i}}{4}\,(\bar{\mathcal{D}}_{\dot{\alpha}\underline{1}}\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} + 4\bar{\mathcal{S}}_{\underline{11}})\,\mathcal{V}_1\,,\\
\mathcal{W} &= \frac{1}{8\pi}\oint\mathrm{d}\zeta(\bar{\mathcal{D}}_{\dot{\alpha}\underline{2}}\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}} + 4\bar{\mathcal{S}}_{\underline{22}})\mathcal{V}(\zeta) = \frac{\mathrm{i}}{4}\,(\bar{\mathcal{D}}_{\dot{\alpha}\underline{2}}\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}} + 4\bar{\mathcal{S}}_{\underline{22}})\,\mathcal{V}_{-1}\ .
\end{aligned} \tag{B.8}$$

The field strength (B.6a) can be shown to be invariant under gauge transformations of the form

$$\delta\mathcal{V} = \lambda + \widetilde{\lambda}\,, \tag{B.9}$$

with the gauge parameter $\lambda(z, u^+)$ being a covariant weight-zero arctic multiplet, and $\widetilde{\lambda}$ its smile-conjugate,

$$\mathcal{D}^+_\alpha \lambda = \bar{\mathcal{D}}^+_{\dot\alpha} \lambda = 0\,, \qquad \lambda(z, u^+) = \lambda(z, \zeta) = \sum_{k=0}^{+\infty} \zeta^k \lambda_k(z)\,, \tag{B.10a}$$

$$\mathcal{D}^+_\alpha \widetilde{\lambda} = \bar{\mathcal{D}}^+_{\dot\alpha} \widetilde{\lambda} = 0\,, \qquad \widetilde{\lambda}(z, u^+) = \widetilde{\lambda}(z, \zeta) = \sum_{k=0}^{+\infty} (-1)^k \zeta^{-k} \bar{\lambda}_k(\zeta)\ . \tag{B.10b}$$

To prove the gauge invariance of $\mathcal{W}$, the only non-trivial observation required is that the constraints on λ and $\widetilde{\lambda}$ imply

$$\left(\bar{\mathcal{D}}_{\dot\alpha\underline{1}} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} + 4\bar{\mathcal{S}}_{\underline{11}}\right) \lambda_1 = 0\,, \qquad \left(\bar{\mathcal{D}}_{\dot\alpha\underline{2}} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} + 4\bar{\mathcal{S}}_{\underline{22}}\right) \bar{\lambda}_1 = 0\ . \tag{B.11}$$

It can also be demonstrated that the following super-Weyl transformation of the gauge prepotential $\mathcal{V}(z, u^+)$,

$$\delta_\sigma \mathcal{V} = 0\,, \tag{B.12}$$

implies the super-Weyl transformation of $\mathcal{W}$, eq. (B.3).

C. $\mathcal{N} = 1$ AdS Killing supervectors

The covariant derivatives of the $\mathcal{N} = 1$ anti-de Sitter superspace $\mathrm{AdS}^{4|4}$,

$$\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha}) = E_A{}^M \partial_M + \frac{1}{2}\phi_A{}^{bc} M_{bc}\,, \tag{C.1}$$

obey the following (anti-)commutation relations:

$$\{\nabla_\alpha, \nabla_\beta\} = -4\bar\mu M_{\alpha\beta}\,, \qquad \{\nabla_\alpha, \bar\nabla^{\dot\beta}\} = -2\mathrm{i}(\sigma^c)_\alpha{}^{\dot\beta} \nabla_c\,, \tag{C.2a}$$

$$[\nabla_a, \nabla_\beta] = -\frac{\mathrm{i}}{2}\bar\mu(\sigma_a)_{\beta\dot\gamma} \bar\nabla^{\dot\gamma}\,, \qquad [\nabla_a, \nabla_b] = -|\mu|^2 M_{ab}\,, \tag{C.2b}$$

with μ a complex non-vanishing parameter which can be viewed to be a square root of the curvature of the anti-de Sitter space, see, e.g., [7] for more detail. The symmetries of $\mathrm{AdS}^{4|4}$ are generated by the corresponding Killing supervectors defined as

$$\Lambda = \lambda^a \nabla_a + \lambda^\alpha \nabla_\alpha + \bar\lambda_{\dot\alpha} \bar\nabla^{\dot\alpha}\,, \qquad [\Lambda + \omega^{bc} M_{bc}, \nabla_A] = 0\,, \tag{C.3}$$

for some local Lorentz transformation associated with ω^{bc}. As shown in [7], the equations in (C.3) are equivalent to

$$\omega_{\alpha\beta} = \nabla_\alpha \lambda_\beta\,, \qquad \nabla_\alpha \lambda^\alpha = 0\,, \qquad 0 = \frac{\mathrm{i}}{2}\mu\lambda_{\alpha\dot\alpha} + \bar\nabla_{\dot\alpha}\lambda_\alpha\,, \tag{C.4}$$

$$0 = \nabla_{(\alpha}\lambda_{\beta)\dot\beta}\,, \qquad 0 = \bar\nabla^{\dot\beta}\lambda_{\alpha\dot\beta} + 8\mathrm{i}\lambda_\alpha\ . \tag{C.5}$$

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D. Stereographic projection for AdS spaces

Consider a d-dimensional anti-de Sitter space AdS_d. It can be realized as a hypersurface in $\mathbb{R}^{d-1,2}$ parametrized by Cartesian coordinates $Z^{\hat{a}} = (Z^d, Z^a)$, with $a = 0, 1, \ldots, d-1$. The hypersurface looks like

$$-(Z^d)^2 - (Z^0)^2 + \sum_{i=1}^{d-1} (Z^i)^2 = -(Z^d)^2 + Z^a Z_a = -R^2 = \text{const} . \tag{D.1}$$

One can introduce unconstrained local coordinates for AdS_d as a natural generalization of the stereographic projection for S^d. Let us cover AdS_d by two charts:

(i) the north chart in which $Z^d > -R$; and

(ii) the south chart in which $Z^d < R$.

Given a point $Z^{\hat{a}}$ in the north chart, its local coordinates x^a will be chosen to correspond to the intersection of the plane $Z^d = 0$ and the straight line connecting $Z^{\hat{a}}$ and the "north pole" $Z^{\hat{a}}_{\text{north}} = (-R, 0, \ldots, 0)$. Similarly, given a point $Z^{\hat{a}}$ in the south chart, its local coordinates y^a will be chosen to correspond to the intersection of the plane $Z^d = 0$ with the straight line connecting $Z^{\hat{a}}$ and the "south pole" $Z^{\hat{a}}_{\text{south}} = (R, 0, \ldots, 0)$.

In the north chart, one finds

$$x^a = \frac{R\, Z^a}{R + Z^d} , \qquad x^a x_a < R^2 . \tag{D.2}$$

A short calculation for the induced metric, $\mathrm{d}s^2 = -(\mathrm{d}Z^d)^2 + \mathrm{d}Z^a\, \mathrm{d}Z_a$, gives the conformally flat form:

$$\mathrm{d}s^2 = \frac{4\, \mathrm{d}x^a \mathrm{d}x_a}{\big(1 - R^{-2}\, x^2\big)^2} , \qquad x^2 = x^b x_b . \tag{D.3}$$

In the south chart, one similarly gets

$$y^a = \frac{R\, Z^a}{R - Z^d} , \qquad y^a y_a < R^2 . \tag{D.4}$$

The metric is obtained from (D.3) by replacing $x^a \to y^a$.

In the intersection of the two charts, the transition functions are:

$$y^a = -R^2 \, \frac{x^a}{x^2} . \tag{D.5}$$

This is an inversion, that is, a discrete conformal transformation.

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